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vii

William C. Sibley.

ELEMENTS

OF

ARITHMETIC,

TRANSLATED FROM THE FRENCH

OF

M. BEZOUT,

AND ADAPTED TO THE USE

OF

AMERICAN SCHOOLS.

By NATHANIEL HAYNES, A. B.
Tutor in Mathematics at Gardiner Lyceum.

Hallowell:
PRINTED BY GLAZIER AND CO.
1824.

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37
.B573
1824

DISTRICT OF MAINE,—ss.

[L. S.] BE IT REMEMBERED, That, on this thirty-first day of May, in the year of our Lord one thousand eight hundred and twenty-four, and the forty-eighth year of the Independence of the United States of America, Mr. NATHANIEL HAYNES, of the District of Maine, has deposited in this office, the title of a Book, the right whereof he claims as Proprietor, in the words following, viz:

> "Elements of Arithmetic, translated from the French of M. Bezout, and adapted to the use of American Schools; by Nathaniel Haynes, A. B. Tutor in Mathematics at Gardiner Lyceum. Hallowell: Printed by Glazier & Co. 1824."

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ADVERTISEMENT.

This Arithmetic has been chiefly extracted, from the first part of a Course of Mathematics, composed by Etienne Bezout, an eminent French Mathematician, for the use of the Guards of the Marine and the Pavilion, and afterwards adopted by the Polytechnic School in France. The great excellence of Bezout consists in uncommon perspicuity in the explanation of first principles. Aware that the rigorous forms of Algebraic demonstration are unintelligible to *beginners*, he has followed the more rational method of developing the Elements of Arithmetic, by plain illustrations, and a clear exposition of the reason of its operations. But although many obstacles are removed that have hitherto impeded the progress of the student, he must not expect to become an expert Arithmetician without close application and frequent exercise of his reasoning faculties. To assist him in these efforts, and to render his investigations practically as well as theoretically useful, numerous select Examples have been introduced, showing the application of abstract principles to a great variety of questions in business.

NATH'L HAYNES.

Gardiner, June 1824.

EXPLANATION OF CHARACTERS.

Numbers found in a parenthesis refer to articles, where the proposition is demonstrated that is there relied upon.

= Signifies equality, or equal to; as 100 cents=one dollar; that is, 100 cents are equal to one dollar.

+ Signifies plus, or Addition; as 50+50=100; that is, 50 added to 50 is equal to 100.

— Signifies minus, or Subtraction; as 6—2=4; that is, 6 less 2 is equal to 4.

× Signifies Multiplication; as 4×4=16; that is, 4 multiplied by 4 is equal to 16.

· When placed over a figure, shows that it does not form a part of the true result.

· Signifies, when placed between two figures, their Arithmetical Ratio; thus 15 · 8 signifies the arithmetical ratio of 15 to 8.

: Signifies the Geometrical Ratio between two numbers; thus 12 : 3, signifies the geometrical ratio of 12 to 3.

∺ Signifies a Continued Proportion, which is Arithmetical, or Geometrical, according as the Character between the Terms denotes; thus, ∺ 3 · 7 · 11 signifies an Arithmetical, and ∺ 5 : 20 : 80, a Geometrical, Continued Proportion.

÷ Signifies an Arithmetical, or Geometrical Progression, according as the Character between the Terms denotes; thus, ÷ 2 : 4 : 8 : 16 : &c. signifies a Geometrical progression, and ÷ 3 · 5 · 7 · 9, &c. an Arithmetical progression.

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(*) These Articles and most of the practical examples have been compiled from American Authorities.

RECOMMENDATIONS.

From the Committee, *appointed by the Trustees of Gardiner Lyceum to examine this Arithmetic, previous to its introduction as a Text-Book at the Lyceum.*

Having examined, with some attention, the Arithmetic of Bezout, we do not hesitate to say, that, in our opinion, it contains a very lucid explanation of the principles of this important part of Mathematics, and we hope that its introduction to our Schools will make the study of Arithmetic something more than a mere exertion of memory in committing rules, and a mechanical conformity to them.

The translation is faithfully and correctly made, and the Translator has added many examples to the original work, which, although we have not examined them we hope, will be found to enhance its value.

BENJAMIN HALE, *Principal of Gardiner Lyceum.*
R. H. GARDINER, } *Committee of Trustees.*
JOHN MERRICK, }
Gardiner, Nov. 1824.

Extract of a letter to the Translator, from Parker Cleaveland, L. L. D. **Professor of Mathematics and Natural Philosophy** *in Bowdoin College.*

To Nathaniel Haynes, A. B.

Sir,—I am happy to learn that you are about giving to the Public a translation of the *excellent* Arithmetic of Bezout's, accompanied by "numerous select Examples, showing the application of abstract principles to a great variety of questions in business." It has long been perceived, that most of the Elementary treatises on Arithmetic are very deficient, by not containing rigorous and perspicuous demonstrations of the common rules of Arithmetic. The progress of the student in Mathematics is hereby much retarded, or he is even made to dislike the science. Nothing less rigorous than Bezout will supply the deficiency.

I am Sir, Yours, &c.
P. CLEAVELAND.

Brunswick, Nov. 1, 1824.

ELEMENTS OF ARITHMETIC.

PRELIMINARY NOTIONS CONCERNING THE NATURE AND DIFFERENT KINDS OF NUMBERS.

1. In general, whatever is susceptible of augmentation or diminution is called quantity. Extension, duration, and weight are quantities. Mathematics have for their object all kinds of quantities; but Arithmetic, which is a part of Mathematics, considers only such quantities as are expressed in numbers.

2. Arithmetic is then the science of numbers: it considers their nature and properties; and its design is to teach the easiest method of representing, as well as of compounding and resolving them.

3. In order to form a correct idea of numbers, it is essential to understand what is meant by *unity*.

4. An unit is a quantity, taken for the most part arbitrarily, to serve as a standard of comparison to all quantities of the same kind. Thus a body is said to weigh four pounds; and the pound is the unit or quantity with which the weight of the body is compared. With equal propriety the ounce might have been assumed as the standard of comparison; and then the weight of the body would have been expressed by the number sixty-four.

5. A number expresses the amount of units or parts of an unit in any given quantity. If it is expressed by entire units, the number expressing it is called a *whole number;* if it is composed of entire units and parts of an unit, or only of parts of an unit, the number is called *fractional*, or simply a *fraction.* Three and a half is a *fractional* number; three-fourths form a *fraction.*

6. A number expressed without designating the kind of units, as when we say three or four, three times, four

times, &c., is called an *abstract number*. But when the kind of units is named, as when we say *four pounds, an hundred* tons, &c., the number is called *concrete*. The other kinds of units will be defined in their proper places.

OF NUMERATION AND DECIMALS.

7. *Numeration is the art of expressing all numbers by a limited quantity of names and characters.* These characters are called *figures*. It is important clearly to develope the manner in which figures represent numbers.

8. The characters used in numeration, and the names of the numbers, which they represent, are the following;

0	1	2	3	4	5	6	7	8	9
cipher,	one,	two,	three,	four,	five,	six,	seven,	eight,	nine.

To express higher numbers by these characters, it is convenient to form, by ten single units, *another unit, called ten;* and to compute by tens in the same manner as by single units; thus, we say once ten, twice ten, three times ten, &c.; or twenty, thirty, &c. to ninety. And to represent these new units, the same figures are employed as in expressing simple units; but they are distinguished by the place which they occupy, being put to the left of the simple units. Thus, to represent the number fifty-four, which is composed of five tens and four units, we write 54. To represent sixty, which contains an exact number of tens and no units, we write 60. The cipher shows that single units are wanting, and causes the figure 6 to express the number of tens. In the same manner numeration is continued to ninety and nine inclusively.

9. We must observe as a peculiarity of this numeration, *that a figure placed immediately at the left of another, or followed by a cipher, represents a number ten times greater than when standing alone.*

10. From ninety and nine, numeration is continued by a similar method to nine hundred and ninety-nine. Of ten tens is composed *another unit, which is called an hundred,* because ten tens make an hundred. After the manner of tens, the hundreds are reckoned up to nine; and are represented by the same figures, but placed to the left of the place of tens. To express eight hundred

and fifty-nine, we write 859; because there are eight hundreds, five tens, and nine units, in the number to be expressed. If eight hundred and nine is the number to be expressed, since it contains eight hundreds, no tens, and nine units, it is written, 809; that is, a cipher is put to occupy the place of tens which are wanting. If units are wanting also, two ciphers are employed: thus; to express eight hundreds, we write, 800.

11. It may be remarked then, that by means of this arrangement, *a figure followed by two others, or by two ciphers, denotes a number an hundred times greater than when written alone.*

12. From nine hundred and ninety-nine, numeration is continued by a like arrangement up to nine thousand, nine hundred and ninety-nine. Of ten hundreds *is formed an unit called a thousand;* because ten times an hundred make a thousand. These are reckoned as the preceding, and represented by the same figures placed to the left of the hundreds. Thus, to signify eight thousand, seven hundred and fifty-nine, we write 8759; to express eight thousand and nine, we write 8009; and for eight thousand, 8000. Hence, *a figure followed by three others, or by three ciphers, denotes a number a thousand times greater than when it is written alone.*

13. And by continuing to include ten units of a certain order in a single unit; and by placing these new units in orders or periods more and more advanced towards the left, all imaginable whole numbers are expressed in an uniform manner, by the aid of ten figures only.

14. To enumerate with facility a number expressed by any given number of figures, it is separated into divisions of three figures each, proceeding from right to left; and each division beginning at the right receives the following names; *units, thousands, millions, billions, trillions, quadrillions, quintillions, sextillions, &c.* The first figure of every division, proceeding always from the right, will have the name of the division, the second that of the tens, and the third that of the hundreds. Thus, in passing from the left, every division is enunciated as if it stood separate by itself; and at the end of each division, the name of the division is pronounced. For example, to reckon the following number, viz.

quadrillions, trillions, billions, millions, thousands, units
23 . . . . 456 . . 789 . . 234 . . . 565 . . 456;

we say twenty-three quadrillions, four hundred and fifty-six trillions, seven hundred and eighty-nine billions, two hundred and thirty-four millions, five hundred and sixty-five thousands, and four hundred and fifty-six units.

15. It results from the numeration which has been explained, and which is altogether arbitrary, that as we advance from right to left, the units of which a number is composed become ten times greater at each removal. Consequently, a number may be rendered ten, an hundred, or a thousand times greater, by putting after it, one, two, three, &c. ciphers. On the contrary, as we pass back from the left to the right, the units become ten times smaller at each removal.

16. Such is numeration. It is the basis of all kinds of computation; although we are not always limited to calculating by ten, by tens of tens, &c.

17. To show the value of quantities smaller than the chosen unit, that unit is divided into smaller units. The number of these divisions is indifferent in itself, provided the quantities can be measured to which we intend to apply them. But the principal object to be kept in view is to render calculation in these cases as convenient as possible; therefore, instead of dividing the unit at once into a great many parts, it is better to separate it into a small number, and afterwards to separate these divisions into others still less. It is thus in the divisions of money; the dollar is divided into ten equal parts called dimes, and each of these into ten others called cents, each of the last being again divided into ten others called mills.

18. A number composed of parts, having relation to each other, but referring to different units, is called a *complex number*; and in contradistinction, that which includes units of only one kind, is called a *simple number*. 8 dollars, or 8 leagues is a simple number; but 8 dollars, 7 cents and 6 mills form a complex number.

19. The principal unit is subdivided in a peculiar manner in every measure. The subdivisions of the mile are different from those of the pound, and those of the pound from those of the day and hour. These divisions will be explained in treating of complex numbers.

20. But of all the divisions and subdivisions of which the unit is susceptible, that made by decimals, that is, by dividing the unit into parts whose value is ten times smaller at each removal from the place of whole numbers, is unquestionably the most convenient. The calculation of decimals, so useful in the practice of mathematics, is the same as for whole numbers. We shall proceed to explain the principle of their formation.

21. To express in decimals, parts smaller than the principal unit, we suppose this unit (of whatever kind it may be, whether the mile, pound, or dollar,) to be composed of ten parts, as 10 is conceived to be composed of so many simple units, or as the pound is supposed to be composed of 16 ounces; these new units, in opposition to tens, are called *tenths;* and they are represented by the same figures as simple units; and since they are ten times smaller than the simple unit, they are placed to the right of that figure which represents simple units. But to prevent error in taking tenths for simple units, it is convenient to fix the place of units by a particular mark; that which is in common use, is a comma (,); which is placed to the right of the figure that represents simple units; or which is the same thing, between the units and tenths. Thus, to denote that twenty-four units and three-tenths are expressed, we write 24,3.

22. The hundredths may be regarded as units which have been formed of ten other units, each ten times smaller than an unit in the place of tenths; and by the same analogy placed to the right of the tenths. These new units, which are ten times smaller than the tenths, will be an hundred times less than the principal unit, and for that reason are called *hundredths.* Thus when we wish to express twenty units, three tenths, and five hundredths, we write 20,35.

23. In like manner let us suppose the ~~hundredth~~ thousandth to be composed of ten parts; these parts will be a thousand times smaller than the principal unit, and are therefore called *thousandths;* and since they are ten times less than the hundredths, they are placed to the right of them. By continuing thus to subdivide by tens, a new series of units is formed, which are named successively, *ten thousandths, hundred thousandths, millionths, ten mil*

lionths, hundred millionths, billionths, &c.; which are placed in divisions more and more removed from the right of the separatrix or comma.

24. The parts of an unit, which have been described, are called *decimals*.

25. The manner of expressing them is the same, as for whole numbers. After having expressed the figures, which are to the left of the separatrix, the decimals, (which are written in the same manner as whole numbers,) are expressed; but at the end, the name of the decimal unit of the last kind is added. Thus, to express the number 34,572, we say thirty four units and five hundred and seventy two thousandths. If the number were leagues, it would be read thirty-four leagues and five hundred and seventy-two thousandths of a league.

The reason for this is easily perceived, if we observe, that in the number 34,572, the figure 5 can be indifferently expressed, either by calling it five tenths, or five hundred thousandths; since the tenth (22) has the value of 10 hundredths, and the hundredth (23) the value of 10 thousandths, the tenth will contain ten times the 10 thousandth, or 100 thousandths; five tenths then have the value of 500 thousandths. For the same reason, the figure [illegible] may be expressed by saying seventy thousandths; since (23) each hundredth has the value of 10 thousandths.

26. The kind of units of which the last figure is composed, may be easily determined by counting each figure successively, from the separatrix towards the right, by the following names; *tenths, hundredths, thousandths, ten thousandths, &c.*

27. If in the sum there are no entire units, but only parts of an unite, a cipher is put in the place of units; thus to express 125 thousandths, we write 0,125. If 25 thousandths are to be expressed, we write 0,025, putting a cipher between the separatrix and the other figures,—as much, to denote a deficiency of tenths, as to give to the following parts their true value. For the same reason, to express 6 thousandths, we write 0,006.

28. We will now examine the changes, that may be made in the value of numbers by changing the place of the separatrix. Since the separatrix determines the places of units and of all other figures, whose value de-

pends upon their distance from it; if the separatrix is advanced one, two, or three places towards the left, the number is rendered 10,100,1000, &c. times smaller; and on the contrary, if the separatrix is removed one, two, or three places towards the right, it becomes 10,100, or 1000 times greater. Thus, having the sum 4327,5264, if we advance the separatrix towards the left and write 432,75264, it is evident, that the thousands of the first number have become the hundreds of the second or new number; the hundreds are the tens, and the tens are the units; the units are the tenths, the tenths are the hundredths, and so on. Each part then has become ten times smaller by the change. On the contrary, by removing the separatrix one place towards the right, the sum is expressed 43275,264; and the thousands of the first number are changed into tens of thousands, the hundreds into thousands, the tens into hundreds, the units into tens, the tenths into units, and so on. And this new number has become ten times greater than the first.

29. This reasoning shows, *that by removing the separatrix two, or three places, towards the left, the number is made* 100, *or* 1000 *times smaller; on the contrary,* 100 *or* 1000 *times greater by removing it two or three places towards the right.*

30. The last observation, which we shall make upon the notation of decimals, is, that their value is not altered by adding any number of ciphers after the last decimal figure. Thus 43,25 is the same as 43,250, or as 43,2500, or 43,25000, &c. Because each hundred has the value of 10 thousandths or 100 ten-thousandths, &c.; and the twenty-five hundredths have the value of 250 thousandths, or 2500 ten-thousandths; and in short, it is the same thing as when we say one hundred cents instead of one dollar.

Let the pupil express the following numbers by figures.

1. Seventy-nine.
2. One hundred and one.
3. Five hundred and twenty-four.
4. Eight hundred and seven.
5. Five tenths.
6. One and eight tenths.
7. Five hundred thousandths.

8. Seven hundred whole numbers and seven hundredths.
9. Nine thousand seven hundred and ten.
10. Ten thousand and four.
11. Fifteen thousand, and fifteen thousandths.
12. One million and one.
13. One hundred millions and one hundred thousand.
14. Five millions, and five hundredths.
15. Two billions, two millions, and two.
16. Four trillions, eight hundred, and five tenths.

ARITHMETICAL OPERATIONS.

31. Addition, Subtraction, Multiplication, and Division, are called the four fundamental operations of Arithmetic. All questions, which depend for their solution upon numbers, are reduced to the practice of some of these rules. It is therefore important they should be thoroughly and familiarly understood.

32. The design of Arithmetic, as has already been explained, is to furnish the most ready method for calculating numbers. The means used for this purpose, consists in reducing the calculation of complicated numbers, to that of numbers more simple, or expressed by the smallest possible number of figures.

The following treatise has this object.

ADDITION OF WHOLE NUMBERS AND DECIMAL PARTS.

33. *Addition teaches to express the total value of many numbers by a single number.* When the numbers, which are proposed to be added, are expressed by one figure, there is no need of a rule; but when they consist of many figures, their total value is found by observing the following method.

Write under each other all the proposed numbers, in such order, that all the unit figures may be in the same vertical column. In like manner place all the tens in the same vertical column; and also the hundreds, thousands, &c. in their respective columns; and draw a line under the whole. Then add all the numbers which are in the column of units. If the sum does not exceed 9, write it *underneath. If it exceed nine, it belongs to the tens. Write underneath the excess of simple units over tens, and count these tens as so many units to be added with the units of the next column. Proceed with the second column as with the first, and continue in the same manner from column to column to the last, under which write the whole amount that is found in it.* This rule is explained by the following examples.

EXAMPLE I.

If 54925 is to be added to 2023; the numbers are written as below.

54925
 2023

56948 sum.

To add these numbers, begin with the units by saying 5 and 3 make 8, which write under the unit column. Then pass to the column of tens and say 2 and 2 are 4; and write 4 under that column. In like manner, at the column of hundreds, 9 and cipher make 9, which is put under its column. And at the column of thousands, 4 and 2 make 6, which is put under its column. And lastly in the column for tens of thousands, we say 5 and cipher make 5, and write it under the same column. The number 56948, found by this operation, is the sum of the two proposed numbers; since it contains the units, tens, hundreds, thousands, and tens of thousands of both successively collected together.

EXAMPLE II.

Required the sum of the four following numbers; 6903, 7854, 953, 7327. They are to be thus written.

6903
7854
 953
7327

23037 sum.

Commencing, as in the first Example, at the right hand column, we say 3 and 4 make 7, and 3 are 10, and 7 make 17; 7 units are written under the first column, and the ten is retained to be joined as an unit to the next column, which is the column of tens.

Passing to the second column, we say 1, which was retained from the first column, and cipher make 1, and 5 make 6, and 5 are 11, and 2 make 13; 3 is written under this column; and for the ten, an unit is retained and added to the following column; where we say 1 and 9 make 10, and 8 make 18, and 9 are 27, and 3 make 30; a cipher is written under this column, and for the three tens of it, three units are retained, and added to the next column, by saying as before, 3 and 6 make 9, and 7 are 16, and 7 make 23; 3 is written under this column, and as there is not another, to which the two remaining tens might be added, they are written in the place of such column. The number 23037 is the sum of the four proposed numbers.

34. *When decimals are to be added; since they are reckoned like other numbers, by tens, in passing from right to left, the same rule is observed as for whole numbers;* units of the same order being placed in the same column. Thus, if it is proposed to add these three numbers, 72,957, 12,8, 124,03; we write them in this manner.

```
 72,957
 12,8
124,03
-------
209,787 sum.
```

This sum is obtained by following the same method as in the preceding cases.

EXAMPLES FOR PRACTICE.

1. What is the sum total of the following numbers; 345674, 98736, 456752, and 7654? Ans. 908816.

2. What is the sum of ,014+,9816+,32+,15914+ 72918+,0047? Ans. 2,20857.

3. I lent a friend 942 dollars, at one time; at another, 741; at another 91 dollars; what is the whole sum lent? Ans. 1774 dollars.

4. What is the sum of one millionth part of an unit added to 9,999999? Ans. 10.

5. A man gave to three sons as follows; to one 3427 dollars; to another 3925 dollars; to another 2947 dollars; what did he give to all of them? Ans. 9399 dollars.

6. A man's taxes were as follows for 3 years; the first 142 dollars, second 165, the third 210 dollars; what did he pay in three years? Ans. 517 dollars.

7. A merchant bought of one person 50 barrels of flour for 300 dollars; of another person, 75 barrels for 525 dollars; and of another person, 125 barrels for 1000 dollars. How many barrels did he buy, and how much did he pay for the whole?

Ans. 250 barrels, and paid 1825 dollars.

8. John, James, and Paul, counting their prize money, John had one thousand, three hundred, and seventy-five dollars; James had three times as much as John; and Paul had as much as both the others; how many dollars had Paul? Ans. 5500 dollars.

SUBTRACTION OF WHOLE NUMBERS AND DECIMAL PARTS.

35. *Subtraction is an operation by which one number is taken from another number.* The result of this operation is called *remainder, excess, or difference.*

To perform subtraction, *the number, which is to be subtracted, is written under the number to be diminished, in the same manner as in addition. A line is then drawn under the whole; and each figure in the lower line is taken from its correspondent figure in the upper line, that is, units are taken from units, tens from tens, hundreds from hundreds, &c. and the remainders are written below in the same order,—and cipher when there is no remainder.*

When the lower figure is greater than its corresponding upper figure, ten units are added to the upper number by borrowing an unit from the next figure on the left, which, in the operation following, must be considered as diminished by an unit.

EXAMPLE I.

It is proposed to subtract 5432 from 8954. The numbers are written in this manner.

8954
5432

3522 remainder.

Commencing this operation by the figure in the place of units, we say, 2 taken from 4, leaves 2 for remainder, which is written under the line in the place of units. Then passing to the tens, we say, 3 from 5, there remains 2, which is written underneath. At the third column we say, 4 from 9, and there remains 5, which is written under its column. Finally at the fourth column, we say, 5 from 8, and 3 remains, which is written underneath; and 3522 is the remainder, which results after subtracting 5432 from 8954.

EXAMPLE II.

We wish to subtract 7987 from 27646.

The numbers are written as follows.

$$\begin{array}{r} 27646 \\ 7987 \\ \hline 19659 \end{array}$$

diff. ~~[illegible]~~

As 7 cannot be taken from 6, we add to 6, ten units, which are obtained by borrowing an unit from 4, the next figure on the left; we then say, 7 from 16, there remains 9, which is written under 7.

Passing to the tens, we do not say 8 taken from 4, but from 3, because the unit which was borrowed has diminished it by 1. And 8 cannot be taken from 3; to 3 we add ten units, which are obtained by borrowing an unit from 6 its neighbouring figure on the left, and then say, 8 from 13, there remains 5, which is written under 8. Passing to the third column, we say in the same manner, 9 from 5, or rather from 15 (borrowing as before); and there remains 6, which is written under 9. At the fourth column, we say for the same reason, 7 taken from 6, or from 16, 9 remains, and is written under 7; and as there is nothing to be subtracted in the fifth column, we write under it, not 2, because 2 has been diminished by the unit which was borrowed from it, but 1; and leave 19659 for the remainder sought.

36. If the figure, from which we wish to borrow, is a cipher, we must not borrow from the cipher, but from the next significant figure; and though it be necessary to borrow 100, 1000, or 10000 units, according as there are

two, three, four, &c. consecutive ciphers; the operation must, nevertheless, be performed on the same principles as in the preceding Examples; that is, 10 only are to be added to the figure for which we borrow; and as this ten is supposed to be taken from the 100, or 1000, &c. which have been borrowed; in order to employ the remaining 90, or 990, the following ciphers are considered as so many nines. The Example annexed will illustrate this operation.

EXAMPLE III.

If from	20064
It is required to subtract	17489
	2575 rem.

We first say, 9 taken from 4, or rather from 14 (by borrowing from the next figure), and there remains 5; then as 8 cannot be taken from 5, and as it is impossible to borrow from the next figure which is a cipher, we borrow an unit from 2, which unit has the value of a thousand compared to the figure upon which we operate. From the thousand which has been borrowed, ten units are taken and added to 5; from 15 we take 8 and 7 remains.

As only 10 units of the thousand borrowed, have been employed, the remaining 990 are used in subtracting the numbers, which correspond to the ciphers; or what amounts to the same thing, each cipher is considered to have the value of 9; thus, we say 4 taken from 9, there remains 5; then 7 from 9, 2 remains; and then, 1 from 1, nothing.

87. If there are decimal parts in the numbers to be operated upon, precisely the same rule is to be followed as in the preceding examples; but to prevent all embarrassment in its application, let the number of decimal figures be made the same in each of the proposed numbers, by placing a sufficient number of ciphers after that number which contains the fewest decimal places; this does not change the value of the number (30)

EXAMPLE IV.

From 5403,25
It is required to take 385,6532

We put two ciphers after the decimals of the upper number: and then operate upon the numbers thus prepared, in the same manner as in the rule given for whole numbers.

```
             5403,2500
              385,6532
             ---------
and find     5017,5968 rem.
```

EXAMPLES FOR PRACTICE.

1. Take one hundred and six from three hundred and twenty-five, what remains? Ans. 219.

2. From ,4562, take ,316; what remains? Ans. ,1402.

3. Charles has thirty-two marbles, and John has twenty-five; how many has Charles more than John? Ans. 7.

4. A person had in bank 9000 pounds, but drew out 1112 pounds; how much money had he remaining in bank? Ans. 7888 pounds.

5. A merchant has an estate in his possession valued at 52394 dollars, he owes 31399 dollars; what is he worth after paying his debts? Ans. 20995 dolls.

6. A borrowed of B 9427 dollars, and paid him at one time 4249 dollars, at another time 1427 dollars, and at another time 947 dollars; what remains due?
Ans. 2804 dolls.

7. From an unit subtract the millionth part of itself; what remains? Ans. ,999999.

8. What is the difference between 67 and ,92?
Ans. 66,08.

9. The revolutionary war in America commenced with Great-Britain, in the year 1775; how many years since, the present year being 1824? Ans. 49 years.

10. Peace between the United States and England took place in 1783; war was again declared in 1812; how long did the peace continue? Ans. 29 years.

11. From Hallowell to the city of New-York is 383 miles. Now if a man should travel 10 days from Hal-

lowell towards New-York, at the rate of 36 miles in each day; how far would he then be from that city?

Ans. 23 miles.

12. If a farmer kill six hogs, which weigh 254, 197, 216, 249, 312, and 363 pounds, and market 1000 weight of the pork; what quantity does he reserve for his own use?

Ans. 591 pounds.

13. A gentleman died leaving an estate of 8423 dollars; which he bequeathed as follows; 2300 dollars to each of his two daughters, and the rest to his son; what was the son's share?

Ans. 3823 dollars.

METHOD OF PROOF FOR ADDITION AND SUBTRACTION.

38. The proof of an arithmetical operation, is another operation, performed to verify the first result.

The proof of addition consists in adding again in parts, but commencing on the left, the sums which have been already added. The amount of the first column is taken from the part, which corresponds to it in the sum found by the first addition.

The remainder is then written underneath and considered as so many tens to be joined to the next figure of the first found sum; from the total of the new sum (formed by the remainder joined to the following figure of the first sum) the amount of the next upper column is subtracted.

The same operation is performed upon every column to the last; the sum total of which being subtracted, there is no remainder if the work is right.

Thus, having found by the [illegible] given for addition the sum of the four following numbers.

$$\begin{array}{r} 6903 \\ 7854 \\ 953 \\ 7327 \\ \hline 23037 \\ \dot{3}\dot{1}\dot{1}\dot{0} \end{array}$$

To prove the truth of the result, we add the same numbers again, beginning at the left, and say 6 and 7 make 13, and 7 are 20, which taken from 23, there remains 3 or 3 tens, which with cipher, the figure following, make 30. We then pass to the second column, and say 9 and 8 make 17, and 9 are 26, and 3 make 29, which taken from 30, there remains 1 or 1 tenth, which joined to 3, the figure following, make 13. We then add the numbers in the third column, by saying 5 and 5 make 10, and 2 are 12, which taken from 13, 1 or 1 tenth remains, which added to 7, the figure following, make 17. The figures in the last column are then added, by saying 3 and 4 make 7, and 3 are 10, and 7 make 17, which taken from 17, nothing remains. We therefore conclude that the first operation was correct.

The evidence of the correctness of the first operation, which this method of proof furnishes, rests on the consideration, that after having successively taken from the sum total all the thousands, hundreds, tens, and units, which composed it, *nothing ought to remain.*

39. *The proof of Subtraction is furnished by adding the remainder, found by the operation, to the number subtracted. If the first operation was correct, the addition of the difference, or remainder, to the number subtracted, will reproduce the number from which the subtraction has been made,* Thus in the third example given in Subtraction, by adding 17489 (the number subtracted) with 256, the remainder, the number 20054, from which the subtraction was made, is reproduced.

$$\begin{array}{r}20054\\17489\\\hline 2565\\\hline 20054\end{array}$$

MISCELLANEOUS EXAMPLES IN ADDITION AND SUBTRACTION.

1. Add together 7861, 345, 8023, Ans. 16229.
2. Add together 66947, 46742 and 132684. Ans. 246373.
. Add together 594321, 76543, 2189, 763, 44.
4. From 16844, take 9786. Ans. 7058.
5. From 103034, take 69845. Ans. 33189.

6. What is the difference between 487,068 and 24,703?
Ans. 462,365

7. How much larger is 380,064 than 87,065?

8. How much smaller is 8,756 than 37,005,078?

9. How much must you add to 7,643 to make 16,487?

10. How much must you subtract from 2,483 to leave 527?

11. A merchant shipped molasses to the amount of $15.000.00; but during a storm the master was obliged to throw overboard to the amount of $853.42; what was the value of the remaining part?

12. Money was first made at Argos eight hundred and ninety-four years before Christ. How long has it been in use at this date, 1824?

13. If I add 500, 627, and 1000, and subtract from their amount 900, what number will remain?
Ans. 1227.

14. A person borrowed at one time 62 dollars; at another time 150 dollars; and at another time, 200 dollars; and afterwards paid 300 dollars. How much did he still owe? Ans. 112 dollars.

15. Subtract 267 from 345 and add 150 to the remainder; what is the sum? Ans. 228.

16. A man had in his desk 1000 dollars. He took out 120 dollars to pay a debt. He afterwards put in 75 dollars. How much was there then in the desk?
Ans. 955 dollars.

17. A ship builder built a ship, which cost him 6241 dollars, and sold her for 5927 dollars; what did he lose?
Ans. 314 dollars.

18. A merchant sent a ship and cargo to sea valued at 11497 dollars, and on her return, she and her cargo were valued at 16579 dollars; what was his gain?
Ans. 5082 dollars.

19. From 943213456, take 744321234; what is the remainder! Ans. 198892222.

20. Add 98764321, 12345678, 82456781, 94321452; what is their sum? Ans. 287888232.

21. An ox was killed whose quarters weighed 642, his hide 105, his tallow 92; what was the weight of the ox? Ans. 869 pounds.

22. A man gave to three sons as follows; to one 3427 dollars; to another 3025 dollars; to another 2947 dollars; how much did he give to all of them?

Ans. 9399 dollars.

23. If the distance from Hallowell to Portland be fifty-six miles; from Portland to Portsmouth fifty-four miles, from Portsmouth to Boston sixty-four miles, from Boston to Hartford ninety-eight miles, thence to New-York one hundred and eleven miles, thence to Philadelphia ninety miles, thence to Baltimore ninety-nine miles, and thence to Washington thirty-eight miles; what is the distance from Hallowell to Washington?

Ans. 610 miles.

24. If the distance from Hallowell to Savannah be 1268 miles, and the distance from Hallowell to Washington 610 miles; how far is Savannah from Washington? Ans. 658 miles.

MULTIPLICATION.

40. MULTIPLICATION *consists in repeating one of two numbers as many times as there are units in the other.* To multiply 4 by 3, is to take the number 4 three times.

41. The number to be multiplied is called the *multiplicand*; the number by which the multiplication is performed, is called *the multiplier*; the result of the operation is called the *product.*

42. The word *product*, in its common acceptation, has a more extensive signification; but in this treatise it will be used to signify the result of multiplication.

The multiplicand and multiplier are also called *factors* of the product; thus, 3 and 4 are called the factors of 12, because 3 times 4 make 12.

43. From the idea which has been given of multiplication, it appears to be an abridged method of performing addition. For if the multiplicand is written under itself as many times as there are units in the multiplier, on adding them, their sum will be equal to the product found by multiplication. Example; instead of multiplying 7

by 3, write 7 three times; as in the following manner, and perform addition.

$$\begin{array}{r} 7 \\ 7 \\ 7 \\ \hline 21 \end{array} \text{ sum.}$$

The sum resulting from this addition is equal to the product of 7 multiplied by 3. But when the multiplier is large, this mode of operation becomes tedious. Multiplication, properly so called, teaches a shorter method for obtaining the same result.

44. When multiplication is considered in an abstract manner without regarding the kind of units employed, the multiplicand and multiplier may be indifferently taken for each other. For example, if 4 is to be multiplied by 3, the product will be 12; whether 4 be multiplied by 3, or 3 by 4; for it is evident that 4 times 3 and 3 times 4 are the same thing. This reasoning may be applied to all other numbers. Therefore when two numbers are to be multiplied, the most convenient may be taken for the multiplier.

47. A product found by the repeated addition of the multiplicand, will contain the same kind of units as the multiplicand. But as the use of the multiplier is to show how many times the multiplicand is to be repeated, it can never be considered as a concrete number.

48. The rules for multiplying numbers composed of several figures, are reduced to the multiplication of one single figure by another.

The learner should accustom himself, by successively adding the same number to itself, to find the product of numbers expressed by a single figure. The following table, which is attributed to Pythagoras, may be used for this purpose.

TABLE OF PYTHAGORAS.

1	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	14	16	18	20	22	24
3	6	9	12	15	18	21	24	27	30	33	36
4	8	12	16	20	24	28	32	36	40	44	48
5	10	15	20	25	30	35	40	45	50	55	60
6	12	18	24	30	36	42	48	54	60	66	72
7	14	21	28	35	42	49	56	63	70	77	84
8	16	24	32	40	48	56	64	72	80	88	96
9	18	27	36	45	54	63	72	81	90	99	108
10	20	30	40	50	60	70	80	90	100	110	120
11	22	33	44	55	66	77	88	99	110	121	132
12	24	36	48	60	72	84	96	108	120	132	144

NOTE. For the benefit of the learner, this Table is enlarged from nine to twelve places of figures.

The first column of the Table is formed by the successive addition of 1 to itself; the second by adding 2 in the same manner; the third, by adding 6; and so of the others.

49. To find, by means of this Table, the product of two numbers which are each expressed by a single figure; one of the numbers, the multiplicand for instance, is found in the upper line; the multiplier is then sought in the first column; and in a line with the multiplier, directly under the multiplicand, the product is found. Thus, to find the product of 9 by 6, I descend from 9, found in the first line, to the space opposite 6 in the first column, and there I find 54, their product.

We now pass to the multiplication of numbers expressed by many figures.

MULTIPLICATION OF A NUMBER EXPRESSED IN MANY FIGURES BY A SINGLE FIGURE.

5. *Write the multiplier, which in this case is supposed to be a single figure, under the unit figure of the multiplicand. Multiply the units of the multiplicand by the multiplier, and if the product be units, write it underneath; but if the product contain tens, write only the units underneath, and reserve the tens to be added, as so many units, with the product of the next multiplication. In the same manner, multiply the tens of the multiplicand, and to their product add the tens, if any, which were reserved from the multiplication of the units. Write under the place of tens in the multiplicand, only the units of this new product; and reserve the tens of it (which are hundreds) to be added as units to the product of the next multiplication. Continue to multiply according to the same rule, all the figures of the multiplicand; the series of figures written underneath will be the product.*

EXAMPLE.

How many feet are there in 2864 fathoms? The fathom is 6 feet. And the question requires that 6 feet should be repeated 2864 times; or, (44) that 2864 feet should be taken 6 times.

We therefore write	2864	for the multiplicand;
and	6	for the multiplier.
	17184	is the product.

To perform this operation, commence with the units by saying, 6 times 4 make 24; write 4 under the place of units, and retain 2 for the two tens.

2dly say, 6 times 6 make 36; and 2 which were retained, make 38; put down 8 and retain 3.

3dly, 6 times 8 make 48, and 3 which were retained, make 51; put down 1 and retain 5.

4thly, 6 times 2 make 12, and 5 which were retained, make 17. This number is written down because the multiplication is finished. The number 17184 is the product required, or the number of feet in 2864 fathoms;

because it contains the 4 units 6 times, the 6 tens 6 times, the 8 hundreds 6 times, and the 2 thousands 6 times, consequently 6 times the number 2864.

MULTIPLICATION BY A NUMBER CONSISTING OF MANY FIGURES.

51. *When the multiplier is expressed by several figures; the same operation which has already been explained, is performed successively with each of its figures, commencing always at the right. Thus, all the figures of the multiplicand are first multiplied by the unit figure of the multiplier, then by the figure in the place of tens, and this second product is written under the first; but as the product of units multiplied by tens is tens,* (44) *the first figure of the second product is written under the place of tens in the first, in order to express its true value.*

The third product, which is obtained by multiplying by hundreds, is placed in like manner under the second, but advanced one place towards the left, because the product of units by hundreds is hundreds. The same rule is followed for all other figures in the multiplier.

After multiplication is performed by all the figures in the multiplier, the several products are added together, and their sum is the whole product.

EXAMPLE.

$$\begin{array}{lr} \text{It is required to multiply} & 65487 \\ \text{by} & 6958 \\ \hline & 523896 \\ & 327435 \\ & 589383 \\ & 392922 \\ \hline & 455658546 \text{ product.} \end{array}$$

We first multiply 65487 by 8, the unit figure of the multiplier, and write successively under the line which was drawn below the multiplier, 523896, the figures of the product found by the rule given for the first case (50).

The number 65487 is multiplied in like manner by 5, the second figure of the multiplier; and the product 327435 is written under the first product, but having its first figure 5 under the tens of the first product.

Multiplying 65487 by the third figure 9, the product, 589383 is written under the preceding, but having its first figure 3 put in the place of hundreds, because 9, the multiplying figure, has the value of hundreds.

Finally 65487 is multiplied by 6 the last figure of the multiplier, and 392922, the product, is written under the preceding product, but with its first figure in the place of thousands; because the multiplying figure expresses thousands. These several products are then added together, and 455658546 is the product of 65487 multiplied by 6958, or the value of 65487 taken 6958 times. Because the number 65487 was taken 8 times by the first operation, 50 times by the second, 900 times by the third, and 6000 times by the fourth.

EXAMPLES FOR PRACTICE.

1. Multiply 847483567 by 768.
Ans. 374387379456.

2. Multiply 264648436 by 3639604.
Ans. 963215506259344.

3. What will 587 firkins of butter come to at 7 dollars per firkin? Ans. 4771 dollars.

4. What will be the worth of 924 tons of potash, if 1 ton sell for 95 dollars? Ans. 87780 dollars.

5. There is an orchard containing 9 rows of trees, and there are 57 trees in each row; how many trees are there in the orchard? Ans. 513.

6. A man bought 9 pieces of broadcloth, each piece containing 47 yards, at 6 dollars a yard; and 25 barrels of flour at 7 dollars a barrel; what did he give for the whole? Ans. 2713 dolls.

52. *When the multiplicand or the multiplier or both, are terminated by ciphers, the operation may be abridged by neglecting the ciphers in the multiplication, but placing all of them afterwards at the right of the product.*

EXAMPLE.

It is proposed to multiply 6500
by 350

325
195

2275000

This operation is performed by neglecting the ciphers, multiplying 65 by 35, and placing at the right of their product as many ciphers as were contained in both the factors.

The multiplicand, 6500, truly represents the value of 65 hundreds; therefore when 65 is multiplied, the product is understood to be hundreds. In like manner, the multiplier, 350, denotes 35 tens; when multiplication is made by 35, the product will be understood to be tens; the whole product will then have the value of tens of hundreds, that is, of thousands; three ciphers therefore must be added to the right. This reasoning applies to all cases, in which one, or both factors are terminated by ciphers.

53. *When ciphers are found between the figures of the multiplier, we may dispense with writing them in the product, and pass to the next figure. But the product, which arises from multiplication by this next figure, must be advanced towards the left one place more than the number of ciphers in the multiplier, that is, two places if there is one cipher; three if there are two; and so on.*

EXAMPLE.

If - - - - 42052 be multiplied
by - - - - 3006

252312
126156

126408312 is the product.

In this operation, after having multiplied by 6, and written down the product 252312, we next multiply by 3, because multiplication by a cipher gives only ciphers for the product; but the product 126156 must be written in such a manner as to express thousands; its first figure on the right will therefore be removed three places towards the left, that is, one place more than the number of ciphers interposed between the figures of the multiplier.

EXAMPLES FOR PRACTICE IN THE TWO LAST CASES.

1. Multiply 35012 by 100.	Ans. 3501200.
2. Multiply 638427 by 500.	Ans. 319213500.
3. Multiply 2107900 by 70.	Ans. 147553000.
4. Multiply 9120400 by 90.	Ans. 820836000.
5. Multiply 7854 by 350.	Ans. 27489000.
6. Multiply 62500 by 520.	Ans. 32500000.

MULTIPLICATION OF DECIMAL PARTS.

54. The same rule is observed for the multiplication of decimal parts as for whole numbers. In the operation the separatrix is neglected, but in the product as many figures are pointed off on the right, as there are decimal places in both the factors.

It is proposed to multiply 54,23
by 8, 3

$$\begin{array}{r} 16269 \\ 43384 \\ \hline 450,109 \end{array}$$

We multiply 5423 by 83 and the product is 450109, but as there are two decimal places in the multiplicand and one in the multiplier, three figures are separated upon the right of the product, which becomes 450,109, such as it should be.

The reason for this rule is easily understood by observing, that if the multiplier were 83, the product in decimals would be *hundredths,* because the multiplicand 54,23 would have been repeated 83 times, the decimals of which are hundredths. But as the multiplier is 8, 3, that is, (21,) ten times smaller than 83, the product must contain units ten times smaller than *hundredths;* the last figure in the product must then be *thousandths.* And there must be three decimal figures in the product, because there are as many in the multiplicand and multiplier. The same reasoning applies to all multiplications of decimals.

EXAMPLE II.

If - - - - 0,12 are to be multiplied.
by - - - - 0,3

0,036 is the answer.

We first multiply 12 by 3, which gives 36 for the product. The rule directs to point off three figures for decimals , but there are only two in the product. By applying the reasoning of the preceding case, this embarrassment is easily removed, for it requires the interposition of a cipher between 36 and the separatrix. Because if 0,12 is multiplied by 3, the product is evidently 0,36 ; but since we multiply by 0,3, that is, by a number ten times smaller than 3, the product must be ten times smaller than 0,36, or it will be thousands ; and this value is expressed by writing it (28) 0,036.

EXAMPLES FOR PRACTICE IN THE MULTIPLICATION OF DECIMALS.

1. Multiply 223,86 by 2,500. Ans. 559,65000.
2. Multiply 35,640 by 26,18. Ans. 933,05520.
3. Multiply 8,4960 by 2,618. Ans. 22,2425280.
4. Multiply 0,5236 by 0,2808. Ans. 0,14702688.
5. Multiply 0,11785 by 0,27. Ans. 0,0318195.

55. As the ordinary use of decimals is to facilitate calculation, by substituting instead of an exact computation a sufficient and ready approximation ; a rule is here given for abridging the operation when the product is required only to a certain degree of exactness.

For example, if 45,625957 is to be multiplied by 28,635, and the decimal part of the product is required only to thousandths of an unit ; the numbers are written as below ; that is, after having reversed the order of the figures in one of the factors, we write it under the other, and make its unit figure correspond to the decimal figure in the multiplicand, which is removed two places towards the right from the figure, which has the value of that to which we intend to limit the product. Multiplication is then performed, by neglecting, in the multiplicand, all the figures to the right of that by which we multiply; and the products found by multiplying by the several figures of the multiplier, are written vertically under each other. These products are then added together and two figures suppressed on the right; but the last of the remaining figures is augmented by unity, when the two suppressed figures exceed 50. The separatrix is then put in the place required by the kind of decimals proposed.

EXAMPLE.

We wish to multiply 45,625957
by 28,635
but do not require the decimal part of the product nearer than to thousandths of an unit.

The numbers are written in this manner.

45,625957
53682

91251914
36500760
2737554
136875
22810

130649913
Product 1306,499

If multiplication had been made in the ordinary manner, the product would be 1306,499278695, which corresponds with the preceding up to the third decimal, as was required.

When there are not decimal places enough in the multiplicand to allow the unit figure in the multiplier to correspond to the figure which the rule requires, ciphers may be put to supply the deficiency.

USES OF MULTIPLICATION.

56. We propose to show some uses to which multiplication may be applied, by furnishing such examples as will direct the learner in solving questions of a similar nature.

In general, *multiplication is useful for finding the total value of many units, when the value of each separate unit is known.* For example, how much will 5842 acres of land come to, at 54 dollars an acre? 54 dollars must be multiplied by 5842, or (44) 5842 dollars by 54; and 315468 dollars will be the whole price demanded.

The following are the Tables of Units referred to in the preceding examples.

SQUARE MEASURE.

144	square inches	make	1 square foot.
9	square feet		1 square yard.
30¼ 272¼	square yards or square feet		1 square rod, perch, or pole.
40	square rods		1 rood.
4	roods		1 acre.

FEDERAL MONEY.

10	mills (m.)	make	1 cent,	marked c.
10	cents		1 dime,	d.
10	dimes		1 dollar,	dol. or $.
10	dollars		1 eagle,	E.

The denominations in these Tables will be subsequently recurred to.

Example 2. How much will 5954 cubic feet of water weigh, 72 pounds being the weight of one foot? 72 pounds must be multiplied by 5954, or 5954 by 72; and 428688 pounds will be the weight of 5954 cubic feet.

IN CUBIC OR SOLID MEASURE,

1728	solid inches	make	1 solid foot.
128	solid feet		1 cord of wood.
40 50	solid feet of round timber or solid feet of hewn timber		1 ton or load.

IN AVOIRDUPOIS WEIGHT,

16 drams (dr.)	make	1 ounce, marked	oz.
16 ounces		1 pound,	lb.
28 pounds		1 quarter,	qr.
4 quarters		1 hundred weight,	cwt.
20 cwt.		1 ton,	T.

By this weight are weighed all heavy goods, except gold, silver, corn, bread, and liquors, which are weighed by Troy weight.

57. *Multiplication may be used for reducing units of one denomination into units of a lower denomination.*—For example; dollars may be reduced to dimes, cents, &c.; and tons to pounds, ounces, &c. *by multiplying the units of the higher denomination by the number of the next lower which expresses an unit in this higher denomination, and adding to the product the units, if there be any, of the same name.* Thus, if the number of pence in 8 pounds, 17 shillings and 7 pence is required; we seek in the following Table for the number which expresses the shillings in a pound, and find that it is 20.

ENGLISH MONEY.

4 farthings	make	1 penny.	£ denotes pounds.
12 pence		1 shilling.	s shillings.
20 shillings		1 pound.	d pence.
			q quarters or farthings.

The numbers are then written in the following manner.

$$\begin{array}{lll} £ & s & d \\ 8 & 17 & 7 \\ 20 & & \\ \hline 177\text{ s} & & \\ 12 & & \\ \hline 2131\text{ d} & & \end{array}$$

And the operation is commenced by multiplying £8 by 20 (52,) which gives 160s. for the product, to which add the 17s. in the sum, and the whole number of shillings becomes 177; which is multiplied by 12, because 12 pence make 1 shilling, and the product is 2124 pence, to which add the 7 pence in the sum, and 2131 is the whole number of pence contained in £8 17s. 7d.

MEASURE OF TIME.

60 seconds (s. or ″) make	1	minute, marked	m. or ′.
60 minutes	1	hour,	h. or °.
24 hours	1	day,	d.
7 days	1	week,	w.
4 weeks	1	month,	m.
12 months, 1 day 6 hours, or 365 days and 6 hours,	1	Julian year,	Y.

If the number of minutes in a common year of 365 days, 5 hours, 48 minutes, is required; these numbers are written in this manner,

```
 D.   h.   m.
 365   5   48
  24
 ----
 1465
 730
 ----
 8765
   60
 ------
 525948
```

Because 24 hours make 1 day, 24h. are multiplied by 365 and the product is 8760h., to which the 5h. are added, and 8765, the product, is multiplied by 60 (52,) because the hour contains 60 minutes, and 525900 m. is the product, to which add 48m. and 525948 will be the number of minutes contained in the common year.

TROY WEIGHT.

24 grains (grs.) make	1	penny weight, marked	dwt.
20 dwt.	1	ounce,	oz.
12 oz.	1	pound,	℔.

By this weight are sold gold, silver, corn, bread, and liquors.

APOTHECARIES' WEIGHT.

20 grains (gr.)	make	1 scruple,	marked	sc.
3 scruples		1 dram,		dr. or ʒ.
8 drams		1 ounce,		oz. or ℥.
12 ounces		1 pound,		℔.

Apothecaries compound their medicines by this weight, but buy and sell their drugs by Avoirdupois Weight.

DRY MEASURE.

2 pints (pts.)	make	1 quart,	marked	qts.
2 quarts		1 pottle		pot.
2 pottles		1 gallon,		gal.
2 gallons		1 peck,		pk.
4 pecks		1 bushel,		bu.
2 bushels		1 strike,		str.
8 bushels		1 quarter,		qr.
5 quarters		1 wey or load,		wey.
4 bushels		1 coom,		co.
2 cooms		1 quarter or seam.		
6 seams		1 wey.		
1⅔ weys		1 last,		L.

Note. The diameter of a bushel measure is 18½ inches, and its depth 8 inches. One gallon dry measure contains 268⅘ cubic inches. By this measure, salt, ore, corn, and other dry goods are measured.

ALE AND BEER MEASURE.

2 pints (pts.)	make	1 quart,	marked	qts.
4 quarts		1 gallon,		gal.
8 gallons		1 firkin of Ale,		fir.
9 gallons		1 firkin of Beer,		fir.
2 firkins		1 kilderkin,		kil.
2 kilderkins		1 barrel,		bar.
3 kilderkins		1 hogshead,		hhd.
3 barrels		1 butt,		butt.

The ale gallon contains 282 cubic inches.

WINE MEASURE.

2 pints (pts.) make	1 quart, marked	qts.
4 quarts	1 gallon,	gal.
42 gallons	1 tierce,	tier.
63 gallons	1 hogshead,	hhd.
84 gallons	1 puncheon,	pun.
2 hogsheads	1 pipe or butt,	p. or b.
2 pipes	1 tun,	T.
18 gallons	1 rundlet,	run.
31½ gallons	1 barrel,	bar.

By this measure are measured brandy, spirits, cider, vinegar, and oil. 231 cubic inches make a gallon.

CLOTH MEASURE.

2¼ inches (in.) make	1 nail, marked	na.
4 nails	1 quarter,	qrs.
4 quarters	1 yard,	yds.
3 quarters	1 ell Flemish,	ell Fl.
5 quarters	1 ell English,	ell Eng.
6 quarters	1 ell French,	ell Fr.

LONG MEASURE.

3 barley corns make	1 inch, marked	in.
12 inches	1 foot,	ft.
3 feet	1 yard,	yd.
6 feet	1 fathom, or toise,	fath. to.
5½ yards	1 pole,	pol.
40 poles	1 furlong,	fur.
8 furlongs	1 mile,	mls.
3 miles	1 league,	L.
60 geographical miles, or 69½ statute miles,	1 degree nearly,	deg. or °.

360°, the circumference of the earth,

Also, 4 inches make	1 hand.
5 feet	1 geometrical space.
6 points	1 line.
12 lines	1 inch.

LINEAR MEASURE.

12	points	make	1	line, marked	l.
12	lines		1	inch,	in.
12	inches		1	foot,	ft
6	feet		1	toise, or fathom,	tor. fa.

By following a method similar to those already given, the different denominations of these several weights and measures may be reduced to their lowest terms. If it were required to find how many grains there are in 2lb 4oz. 17 dwt. 5 grs. Troy, we should proceed thus;

```
  lb oz. dwt. grs.
   2  4   17   5
  12
 ---
  24
   4
 ---
  28
  20
 ---
 560
  17
 ---
 577
  24
 ----
 2308
1154
-----
13848
    5
-----
13853 Ans.
```

It may be well to observe that to *double*, *triple*, *quadruple*, &c. signify the same as to multiply by 2, 3, 4, &c.

QUESTIONS FOR PRACTICE.

FEDERAL MONEY.

1. In 40 dollars how many dimes? Ans. 400.
2. In 50 dollars how many cents? Ans. 5000.
3. In 70 dollars how many mills? Ans. 70000.

ENGLISH MONEY.

4. In 3694 shillings, how many pence? Ans. 44328.
5. How many farthings in 69217 pence? Ans. 276868.
6. How many farthings in 6942 pounds? Ans. 6664320.

TROY WEIGHT.

7. How many grains in 47lb. 10 oz. of gold? Ans. 275520.
8. How many grains in 18lb. 5oz. 9dwt. 21gr.? Ans. 106317.
9. What is the value of a silver cup weighing 9oz. 4 dwt. 16 grs. at 3 mills per grain? Ans. $13,29,6.

AVOIRDUPOIS WEIGHT.

10. In 39cwt. 2qr. 14lb. 8oz. 4drs. how many drams? Ans. 1136260.
11. In 8 tons how many ounces? Ans. 286720.
12. How many pounds are in 30 hogsheads of sugar, each weighing 9½ cwt. Ans. 31920.

LONG MEASURE.

13. How many rods in a mile? Ans. 320.
14. How many yards from Boston to Salem, the distance being 18 miles? Ans. 31680.
15. How many barleycorns will reach round the globe? Ans. 4105728000.

CLOTH MEASURE.

16. In 84 yards, how many nails? Ans. 1344.

17. How many yards of Holland in 58 pieces, each containing 36 Ells Flemish? Ans. 1566.

18. At 42 cents a nail, what cost 13 ells Fl. 3 qrs. of broadcloth? Ans. $70,56.

TIME.

19. How many hours in 40 years? Ans. 350400.

20. How many days are there from the 20th of April to the 16th of December following? Ans. 240.

21, How many minutes from the birth of our Saviour to the year 1818? Ans. 955540800.

LAND OR SQUARE MEASURE.

22. In 44 acres, how many perches? Ans, 7040.

23. In 49 acres, 2 rods, 18 poles, how many poles? Ans. 7938.

24. In 34 acres, how many roods and rods?
Ans. { 136 roods.
5440 rods.

WINE MEASURE.

25. How many gallons and pints in 5 tuns of wine?
Ans. { 1260 gals.
10080 pts.

26. How many pints in 5 pipes of wine? Ans. 5040.

27. In 3 hogsheads, how many gills? Ans. 6048.

APOTHECARIES' WEIGHT.

28. In 31lb. 2℥ 6ʒ how many drams? Ans. 2998.

29. In 9℔ 8℥ 1ʒ 2sc. 19grs. how many grains? Ans. 55799.

30. In 14℔ how many grains? Ans. 80640.

DRY MEASURE.

31. In 30 chaldrons of coal, how many bushels?
Ans. 960.

32. In 24 bushels, how many pints? Ans. 1536.

33. At 3 cents a quart, what will 5 bus. 2 pks. 3 qts. of salt come to? Ans. $5,37.

SOLID MEASURE.

34. In 12 tons of round timber, how many cubic inches? Ans. 829440.

35. In 12 tons of hewn timber, how many cubic inches? Ans. 1036800.

36. How many solid inches in a cord of wood?
Ans. 221184.

37. What will 587 firkins of butter come to at 7 dollars per firkin? Ans. 4109 dollars.

38. A man had a farm on which he raised 360 bushels of wheat; and another on which he raised 6 times as much; what quantity did he raise on both of them?
Ans. 2520 bushels.

39. A man had an estate which he divided among 9 sons as follows, viz. to the first eight, he gave 333 dollars each; to the ninth he gave as much wanting 1000 dollars as to the other eight; what was the value of the estate, and also of the ninth son's share?

Ans. { 4328 dollars value of the estate,
1664 dollars ninth son's share.

40. A farmer sheared 364 sheep, six years successively; each sheep yielded 3 pounds of wool per year. How much wool had he yearly; and how much in six years?

Ans. { 1092 pounds yearly,
6552 pounds in 6 years.

41. Sold 342 tons of iron at 142 dollars per ton; what was the price of the whole? Ans. 48564 dollars.

42. In the West Indies 742 thousand of boards were sold at 62 dollars per thousand; what did they come to?
Ans. 46004 dollars.

43. What will 6422 quintals of fish come to at 6 dollars per quintal? Ans. 38532 dollars.

44. A merchant at the expiration of the ten years he had been in trade, found himself worth 13000 pounds. His books showed that for the last three years he had cleared 875 pounds a year; the three preceding, 586 pounds a year; and for the time previous to that, 864 pounds a year. With what sum did he commence business? Ans. £7167.

45. The bridge, which Trajan constructed over the Danube, is said to have had its arches supported by twenty piers. The piers were 60 feet thick, and placed 170 feet distant from each other. How wide was the Danube at that place? and how much did the length of the bridge exceed that of Westminster bridge, which is 1200 feet long? Ans. 4770 feet wide, and 3570 feet longer than Westminster bridge.

46. If a man can travel 7 miles in an hour, how far would he travel in 8 days, when the days are 9 hours long? Ans. 504 miles.

47. If 10 men can do a piece of work in 7 days, how many days will it take one man to do it? Ans. 70 days.

Note. Multiplication by 10, 100, 1000, &c. is performed by annexing 1, 2, 3, &c. ciphers to the multiplicand. See (44) and (52).

48. If 100 men receive 8 dollars apiece, how many dollars do they all receive? Ans. 800 dollars.

49. What would an ox weighing 873 pounds, come to at 10 cents a pound? Ans. $87,30 cts.

SIMPLE DIVISION.

59. *The division of one number by another consists in finding how often the first of two numbers contains the second.*

The number to be divided is called the *dividend;* the number by which division is performed, is called the *divisor;* and that which expresses the number of times the dividend contains the divisor, is called the *quotient.*

It is not always the object of division to know how often one of two proposed numbers contains the other, but the operation in all cases, is performed as if that were the end in view. Hence it follows, that if the divisor be multiplied by the quotient, their product will be the dividend; because the divisor is taken as many times as it is contained in the dividend. And this is a general principle, whether the quotient be a whole number or a fraction.

The kind of units contained in the quotient is not determined by those of the dividend or the divisor; because the dividend and the divisor remaining the same, the quotient will always be numerically the same, although the nature of its units may be very different, according to the nature of the question which occasions the division. For example, if it is required to know how often £8 contains £4, the quotient will be an abstract number which expresses 2 times. But if it is demanded how many rods of wall may be built for £8 at the price of £4 for a rod, the quotient will be 2 rods, which is a concrete number that has no relation either to the units in the dividend or the divisor.

Hence it is evident, that the question, which gives rise to the division, can alone determine the nature of the units in the quotient.

DIVISION OF A NUMBER COMPOSED OF SEVERAL FIGURES, BY A NUMBER EXPRESSED BY ONE FIGURE.

60. The operation about to be described, supposes the learner to be already acquainted with the method of finding how often a number, expressed by one or two figures, contains an other expressed by one figure. This knowledge is acquired when the products of numbers expressed by one figure are known by memory. The Table given in article (48) is used for this purpose. For example, if it is required to know how many times 74 contains 9, the divisor 9 is sought in the upper line, and directly under it, the dividend or number nearest to it, which in this case is 72; then the number 8 standing opposite in the first column is the number of times, or the quotient sought.

We will now explain the method of dividing a number composed of several figures.

Place the divisor at the right of the dividend and draw a line between them; draw a line also under the divisor, below which write the figures of the quotient when they are found.

Take the first figure upon the left of the dividend, or the two first figures, if the first does not contain the divisor. Seek how often the first, or the two first figures, contain the divisor, and write the number of times underneath. Multiply the divisor by this quotient figure, and carry the product under the part of the dividend which has been employed. Subtract this product from the part of the dividend, to which it corresponds; and to the remainder, bring down the next figure of the principal dividend. Perform the same operation upon this second partial dividend, which was performed upon the first; place the quotient, to the right of the quotient figure already found, multiply the divisor by the quotient, write and subtract the product as before. In like manner bring down to the side of the remainder of this division the figure in the dividend following the one already brought down, and continue thus to the last inclusively.

This rule is explained by the following example.

EXAMPLE.

Divide 8769 by 7. The numbers are thus written;

dividend 8769 | 7 divisor.
7 | $1252\frac{5}{7}$ quotient.
17
14
36
35
19
14
5

The operation should be commenced at the left of the dividend by saying, in 8 thousand how many times 7;

but we simply say, in 8 how many times 7? It is contained once. This 1 is naturally one thousand, but the figures which are to come after, will give to it its true value; 1 is therefore written under the divisor. We then multiply 7 the divisor by 1 the quotient, and carry the product 7, under 8 the part of the dividend to be divided; after subtraction, 1 is found for the remainder.

The remainder 1 is the part of 8 which has not been divided, and it has the value of 10 when compared with 7 the figure that follows it; the figure 7 is therefore brought down by the side of 1, and the operation continued by saying, in 17 how many times 7? It is contained twice. The quotient figure 2 is then written to the right of 1, the first quotient figure. As in the preceding operation, the divisor 7 is then multiplied by 2, the last quotient figure; the product 14 is then carried under 17 the partial dividend, and after subtraction from it, there remains 3 for the part which cannot be divided. By the side of the remainder 3, 6 the next figure in the dividend is brought down, and the operation continued by saying, in 36 how many times 7? It is contained 5 times; and 5 is written in the quotient. The divisor 7 is then multiplied by 5, and 35 the product is subtracted from 36 the new partial dividend, and 1 remains. By the side of 1, 9 the last figure in the dividend is brought down, and we say in 19 how many times 7? It is contained twice; and 2 is written in the quotient. The divisor is then multiplied by this last quotient figure, and the product 14 subtracted from 19 the partial dividend, and 5 remains. Hence it appears that 8769 contains 7, the number of times denoted by the quotient, or 1252 times with 5 remainder.

It will suffice for the present to observe, that the remainder is written in the quotient over a line, under which the divisor is written, and pronounced *five sevenths*. An explanation will be given hereafter of this kind of numbers.

61. *If in course of the operation any partial dividend does not contain the divisor, a cipher is written in the quotient; and omitting multiplication, another figure is brought down by the side of the partial dividend, and the division continued.*

EXAMPLE.

Divide 14464 by 8.

```
      |  8
14464 |----
 8    | 1808
 ---
 64
 64
 ----
 0064
   64
   --
   00
```

In this case the two first figures of the dividend are taken, because the first does not contain the divisor. In 14, 8 is contained once, and 1 is written in the quotient. 8 is then multiplied by 1, and the product 8 subtracted from 14, and 6 is the remainder. By the side of 6, 4 the next figure in the dividend is brought down; and the operation continued by saying, in 64 how many times 8? It is contained 8 times; and 8 is written in the quotient. After multiplication, the product 64 is subtracted from 64, and 0 remains; by the side of which, 6 the fourth figure in the dividend is brought down. But as 6 does not contain 8, 0 is written in the quotient, and by the side of 6, the last figure 4 of the dividend is brought down; we then say, in 64 how many times 8? It is contained 8 times. After having written 8 in the quotient and subtracted 64, the product by multiplication, nothing remains. And as there is no remainder, we conclude that 14464 contains 8, 1808 times.

EXAMPLES FOR PRACTICE.

		Quotient.	Rem.
1. Divide 46323 by 9.	Ans.	5147.	
2. Divide 1430400 by 7.		204342,	and 6.
3. Divide 8965462 by 6.		1494243,	4.
4. Divide 3728675 by 8.		466084,	3.

5. Divide 3656 dollars equally among 8 men. Ans. 457 dolls. to each.

6. How much beef at 8 cents per pound can be bought for $1,12? Ans. 14 lbs.

DIVISION BY A NUMBER COMPOSED OF SEVERAL FIGURES.

62. When the divisor contains many figures, the operation is conducted in the following manner.

Take upon the left of the dividend as many figures as are necessary to contain the divisor. Then, instead of seeking, as before, how often this part of the dividend contains the entire divisor, seek how many times the first figure of the divisor is contained in the first figure of the dividend, or in the two first figures, if the first does not contain it, and place this quotient under the divisor as in the former case.

Multiply successively, according to the rule in article (50,) *all the figures of the divisor by the quotient figure, and place the product under the corresponding figures of the partial dividend. Subtract it therefrom, and by the side of the remainder bring down the next figure of the dividend, to continue the operation in the same manner.*

Any embarrassment in the application of this rule will be removed by the following examples.

EXAMPLE I.

Divide 75347 by 53.

$$\begin{array}{r|l}
 & 53 \\ \cline{2-2}
75347 & 1421\frac{34}{53} \\
\underline{53} & \\
223 & \\
\underline{212} & \\
114 & \\
\underline{106} & \\
87 & \\
\underline{53} & \\
34 &
\end{array}$$

In commencing the operation, only the two first figures of the dividend are taken, because they contain the divisor; and instead of saying in 75 how many times

53, we seek how often the 7 tens of 75 contain the 5 tens of 53, that is, how often 7 contains 5. It is contained once; and 1 is written in the quotient. 53 is then multiplied by 1 and the product 53 written under 75. After subtraction there remains 22, by the side of which, 3, the next figure in the dividend is brought down, and the operation continued by saying, in 22 how many times 5? (instead of saying in 223 how many times 53.) It is contained 4 times; and 4 is written in the quotient. The two figures of the divisor are then multiplied successively by 4, and the product 212 is placed under 223, the partial dividend. After subtraction there is 11 for remainder; by the side of which, 4 the next figure of the dividend is brought down; we then say, as before, in 11 how many times 4? It is contained twice, and 2 is written in the quotient. We then multiply 53 by 2 and write 106, the product, under 114 the partial dividend. After subtraction there remains 8, by the side of which, the last figure 7 in the dividend is brought down; 87 is then divided as in the preceding manner, and 1 found for the quotient and 34 for remainder, which is written in the quotient as directed in article (60.)

63. When the operation is performed with exactness, we seek the number of times each partial dividend contains the divisor, but as this process often becomes tedious, it is sufficient, as has been seen, to seek how often the highest part of the dividend contains the highest part of the divisor. The quotient thus found is not always the true one; because, by taking parts, a near approximation only can be made. But this estimated value does not differ widely from the truth, and the ensuing multiplication serves to correct any error in the quotient. Thus, if the partial dividend really contain the divisor three times; but on making trial in the manner directed, if it is supposed to be contained 4 times; when the quotient is multiplied by 4, the product obtained will be greater than the dividend. And this multiplication detects the error, for the divisor has been taken more times than it is really contained in the dividend, consequently subtraction cannot be made. The quotient therefore must be diminished by one, two, three, &c. units, until the product can be subtracted. On the contrary, if the divisor assumed

on trial be too small, the remainder after subtraction will be greater than the divisor; which proves that the divisor is contained again in the dividend; and consequently, that the quotient is too small.

A little practice will teach the learner how to augment or diminish the quotient assumed on the first trial.

EXAMPLE II.

Divide 189492 by 375,

$$\begin{array}{r|l} & 375 \\ \cline{2-2} 189492 & \\ 1875 & 505\frac{117}{375} \\ \cline{1-1} 1992 & \\ 1875 & \\ \cline{1-1} 117 & \end{array}$$

This operation is commenced by taking the four first figures of the dividend, because the three first do not contain the divisor. We then say, in 18 only, how many times 3? It is in fact contained 6 times; but the divisor multiplied by 6 would give a product greater than 1894 the dividend; then 5 only is written in the quotient. We then multiply 375 by 5, and after subtracting the product from 1894, there is 19 for remainder. By the side of 19, the next figure 9 of the dividend is brought down; but as 199 does not contain 375, 0 is put in the quotient, and by the side of 199, the next figure 2 in the dividend is brought down, and 1992 is the next partial dividend; of which we take the two first figures only and say, in 19 how many times 3? It is contained 6 times; but for the same reason as before, 5 only is written in the quotient. After operating as before, 117 is found for remainder.

64. An observation may be here made, that will prevent many useless trials, which the learner is liable to make when the second figure of the divisor is considerably greater than the first. In this case, instead of seeking how often the first figure of the divisor is contained in the corresponding part of the dividend, it is better to seek how often the *first figure augmented by unity*, is contained in the corresponding part of the dividend. This

trial will give a quotient nearer approximated to truth than the first.

EXAMPLE III.

Divide 1832 by 288.

$$\begin{array}{r|l} & 288 \\ 1882 & \overline{} \\ 1728 & 6\frac{104}{288} \\ \cline{1-1} 104 & \end{array}$$

In this example, instead of saying in 18 how many times 2, we say in 18 how many times 3, because 288 is a number much nearer to 300 than to 200. And 6, found by this operation, is the true quotient; but if the other method had been followed, 9 would have been the quotient, and three useless trials must have been made.

EXAMPLES FOR PRACTICE.

1. Divide 2150596750 by 125. Ans. 17204774.
2. Divide 49561776 by 5137. Ans. 9648.
3. The price of a pair of shoes is 2 dollars. How many pair may be had for 56 dollars? Ans. 28.
4. Fifty-four apples are to be divided equally between two boys. How many must each boy have? Ans. 27.
5. Divide 1030603615 by 3215. Ans. 320561.
6. Divide 4917968967 by 2359. Ans. $2084768\frac{1255}{2359}$.
7. Divide 1474 dollars equally among 25 men, 30 women, 17 boys, and 27 girls; after taking out 75 dollars for the payment of a legacy. Ans. $$14\frac{13}{99}$.
8. If 45 horses were sold in the West Indies for 9900 dollars. What was the average price of each? Ans. $220.
9. If a farmer, who has a plantation of 520 acres, buy an adjoining one of 375 acres, and divide the whole into five equal portions; how many acres will there be in each portion? Ans. 179.

METHOD OF ABRIDGING DIVISION.

65. In order to render division more easy to be understood, the learner has been required to write under each partial dividend, the product found on multiplying

the divisor by the quotient. But the operation may be so abridged by subtracting the product of each figure of the divisor immediately after its multiplication, that we may dispense with writing these products. The following example will serve to explain this manner of subtracting which is used when the divisor consists of many figures.

EXAMPLE.

Divide 756984 by 932.

$$\begin{array}{r|l} & 932 \\ \cline{2-2} 756984 & \\ 1138 & 812\tfrac{200}{932} \\ 2064 & \\ \cline{1-1} 200 & \end{array}$$

In this operation we take the four first figures of the dividend, which are necessary to contain the divisor, and find that 75 contains 9, 8 times; 8 is then written in the quotient. But instead of carrying the product of 932 by 8, under the dividend 7569, we first multiply 2 by 8, which gives 16; but as 16 cannot be taken from 9, ten is borrowed from the next figure 6, which joined to 9 gives 19, from which taking 16, there remains 3 which is written underneath.

In keeping account of the ten which was borrowed, the figure 6 is not considered as diminished by the unity, which was taken from it, but the ten is retained to be added as an unit to the following product. Thus, continuing the operation, we say 8 times 3 are 24, and 1 which was retained make 25; but as 25 cannot be taken from 6, we borrow from 5, the next figure in the dividend, two tens, which joined to 6 make 26; from which taking 25, there remains 1, which is written under 6; in this manner account has been kept of the first ten by which 6 was diminished; because an additional ten has been subtracted. Account is kept in the same manner of the two tens, which were borrowed. The operation is then continued by saying, 8 times 9 make 72, and 2 which were borrowed, make 74, which taken from 75, there remains 1.

By the side of the remainder 113, the figure 8 of the dividend is brought down, and the operation continued by saying, in 11 how many times 9? Once; then once 2 is 2, which taken from 8, there remains 6; once 3 makes 3, which taken from 3, 0 remains; once 9 is 9, which taken from 11, there remains 2. The figure 4 is then brought down by the side of 206, the remainder, and we say, in 20 how many times 9? It is contained twice. We then say 2 times 2 make 4, which taken from 4, 0 remains; 2 times 3 make 6, which taken from 6, 0 remains; finally 9 times 2 make 18, which taken from 20, there remains 2.

66. It may happen in the course of these partial divisions, that the dividend contains the divisor more than 9 times; but a higher number than 9 must not be placed in the quotient; for if 10 might be used as a quotient, it is a proof that the quotient found by the preceding operation was false, since the ten found in the actual quotient should have pertained to the former one.

67. If the dividend and the divisor are followed by ciphers, an equal number may be removed from each. Thus, if 8000 is to be divided by 400, we need only divide 80 by 4; for it is evident that 80 hundreds do not contain 4 hundreds oftener than 80 units contain 4 units. When only the divisor contains ciphers, as many figures may be cut off from the dividend as there are ciphers in the divisor, and the operation performed as in the last case; but the figures cut off from the dividend must be added with the remainder, to which the whole divisor is affixed to express the true quotient.

EXAMPLES FOR PRACTICE.

1. Divide 1345680000 by 120000. Ans. 11214
2. Divide 16173 by 320. Ans. $238\frac{13}{320}$.
3. Divide 867894 by 300. Ans. $2892\frac{294}{300}$.
4. Divide 15463420 by 1600. Ans. 9664.
rem. 1020.

THE DIVISION OF DECIMAL PARTS.

68. In order to avoid superfluous distinctions, the operation for the division of decimals is reduced to this single rule.

Put after that of the two proposed numbers, which contains the fewest decimal places, a number of ciphers suffi-

cient to make the number of decimals equal in each. This does not change the value of the number (30.) Suppress the separatrix in both, and perform the operation as for whole numbers. There will be no change to be made in the quotient thus found.

EXAMPLE.

Divide 12,52 by 4,3.

These numbers are written $\begin{array}{r|l} 12,52 & \dfrac{4,3}{} \end{array}$

Or rather - - - $\begin{array}{r|l} 12,52 & \dfrac{4,30}{} \end{array}$

by making the decimal places equal in the dividend and divisor.

By suppressing the separatrix, we have 1252 to be divided by 430; and performing the operation,

$$\begin{array}{r|l} & 430 \\ \cline{2-2} 1252 & \\ 392 & 2\frac{392}{430} \end{array}$$

find 2 for quotient, and 392 for remainder, that is, the quotient is 2 and $\frac{392}{430}$.

But as the use of decimals is to avoid that of ordinary fractions, instead of writing the remainder 392 under the form of a fraction as has been done, the operation is continued as in the following example.

EXAMPLE.

$$\begin{array}{r|l} & 430 \\ \cline{2-2} 1252 & \\ 3920 & 2,9116 \\ \multicolumn{1}{r}{500} & \\ \multicolumn{1}{r}{700} & \\ \multicolumn{1}{r}{2700} & \\ \multicolumn{1}{r}{120} & \end{array}$$

After the quotient is found in whole numbers, which in this case is 2, a cipher is put by the side of the remainder 392, which thus receives a value ten times greater than the true one; division is then continued by 430, and 9 which is the result, is put in the quotient, and its true value shown by placing the separatrix between it and the whole number 2. After multiplication and subtraction, a cipher is put by the side of 50 the remainder, which has the same effect as if two ciphers had been placed at first by the side of the dividend; and by placing 1, the last quotient, after 9, the true value is given to it; for it then expresses hundredths.

The operation may be continued in this manner as far as is thought necessary. By limiting the quotient to two decimal figures, its value is found in hundredths of an unit; by extending it to three figures, the value is found in thousandths of an unit; and so on. For an unit more or less could not be applied to the quotient without rendering it too large or too small.

All remainders after division may be thus reduced to decimals.

We will now explain the reason why the suppression of the separatrix in the dividend and the divisor does not change the value of the quotient, when the number of decimals is the same in each. In the last example, the dividend 12,52, and the divisor 4,30 are in fact 1252 hundredths and 430 hundredths, because the whole numbers have the value of hundreds of hundredths (22.) But it is evident, that 1252 hundredths do not otherwise contain 430 hundredths, than 1252 units contain 430 units; therefore the consideration of the separatrix is useless when the number of decimals is equal in the dividend and the divisor.

EXAMPLES IN THE DIVISION OF DECIMALS.

1. Divide 19,25 by 38,5.	Quotient. ,5.
2. Divide ,1606 by ,44.	3,65.
3. Divide ,1606 by 4,4.	,0365.
4. Divide ,1606 by 44,.	,00365.
5. Divide 9, by ,9.	10,.
6. Divide ,9 by 9.	,1.
7. Divide 234,70525 by 64,25.	3,653.
8. Divide 1,0012 by 0,75.	13,34 and rem.

69. *When the quotient of division is required only to a*

certain degree of exactness, the operation may be abridged by the following method.

Suppress upon the right of the dividend as many figures as there are figures in the divisor, less one, and perform the operation in the ordinary manner. If there is no remainder, put in the quotient as many ciphers as there were figures suppressed in the dividend. But if there is a remainder, continue to divide, not by the same divisor as before, for that is impossible, but by the same divisor after having suppressed the last figure on the right. After this division, divide the new remainder by the preceding divisor, having suppressed the last figure on the right. Continue thus to divide, by suppressing, on each division, a figure upon the right of the divisor.

EXAMPLE.

If the quotient of 8789236487 divided by 64423, is required in whole numbers only, suppress the four last figures to the right of the dividend, and divide 878923 by the proposed divisor 64423.

878923	64423
234693	136430
41424..	6442
2772..	644
196..	64
4..	6

We find, at first, 13 for quotient, and 41424 for remainder; then divide 41424, by the divisor 6442, having suppressed 3, its last figure, and find 6 for quotient. This quotient is then written by the side of 13, the first quotient. The remainder 2722 is then divided by 644, after suppressing the last figure to the right of the first divisor; and the quotient 4 written by the side of 136 the principal quotient. The remainder 196 is then divided by 64, after suppressing another figure in the divisor; the quotient is 3, and remainder, 4. Finally, we divide by 6 and have 0 for remainder. The quotient then of 8789236487 divided by 64423, is 136430, without decimals. The exact quotient however is $136430\frac{6597}{64423}$

The new divisor need not be written each time as in the last example; it is sufficient to separate, in each successive division, a figure at the right in the first divisor.

70. If the remainder of the first division is smaller than the divisor after its last figure is suppressed, a cipher must be put in the quotient; and if it is still smaller, after another of the remaining figures is suppressed in the divisor, a cipher must again be put in the quotient; and so on.

71. If, after commencing the operation by suppressing at the right of the dividend as many figures as the rule requires, the remaining figures do not contain the divisor; we must continue to suppress figures on the right of the divisor till it can be contained in the dividend.

72. The operation will be performed with greater exactness, if, after suppressing a figure in the divisor, the last of the remaining figures is augmented by unity, when the figure suppressed is equal to or greater than 5. In like manner the last of the figures remaining in the dividend should be augmented by unity, when the figures, suppressed agreeably to the direction of the rule, surpass 5,50, or 500, according as there are 1,2,3, &c.

73. From the preceding observations it is easy to learn how the quotient may be obtained to a much greater exactness. If, for example, the quotient is demanded to ten thousandths of an unit, or to the fourth place of decimals; the operation requires as many ciphers to be put after the dividend as there are decimals required in the quotient, Division is then performed according to the preceding method. And when the quotient shall have been found as for whole numbers, as many figures are separated at the right, as there are decimals desired.

EXAMPLE.

If the quotient of 6927 divided by 4532, is required to the fourth place of decimals, four ciphers are put after 6927 ; and the question is to find the quotient, in whole numbers, of 69270000 divided by 4552, that is, according to the preceding rule, to divide 69270 by 4532, as follows.

69270 | 4532
23950 | 15285
1290..453
384..45
24..5

The quotient sought is 1,5285, to the fourth place of decimals.

If there are decimals in the dividend, in the divisor, or in both; they may be reduced to the condition of whole numbers by following the direction given in article (68;) the operation is then performed as in the last example.

By observing the direction in article (71,) any given fraction may be readily reduced by this method to decimals. For example, if it be required to reduce $\frac{4253}{9678}$ to decimals, and to have the value to the third place of decimals; 4253000 must be divided by 9678; which (69) is performed by dividing 253[illegible]78 [illegible] (71) by dividing 4253 by 968, according to [illegible] [illegible] method. The quotient thus found is 439; hence, 0,439 is the value, to the third place of decimals, of $\frac{4253}{9678}$.

74. It may nevertheless happ[illegible] that the quotient found by these rules will not be e[illegible] by 1, 2, or 3, units in the last figure. And although this case rarely occurs, it may not be improper to observe, that error may be prevented by separating at the commencement of the operation, on the right of the dividend, only two figures less than there are in the divisor, and then proceeding as before. When the quotient is found, the last figure in it is suppressed; and to the last remaining figure an unit is added, when the suppressed figure is greater than 5.

EXAMPLES FOR PRACTICE.

1. Reduce $\frac{3}{8}$ to a decimal whose value shall be found in three figures. Ans. ,375.

9. Required the value of $\frac{22}{25}$ to two places of decimals. Ans. ,88.

3. Required the value of $\frac{1}{145}$ to the fifth place of decimals. Ans. ,00689.

4. Reduce $\frac{3}{4}$ to decimals. Ans. ,75.

PROOF OF MULTIPLICATION AND DIVISION.

75. The means of proving these two operations may be derived from the definition which has been given of them.

In multiplication, the multiplicand is taken as many times as there are units in the multiplier. Hence, if we seek how many times the product contains the multiplicand, that is, (59) if we divide the product by the multiplicand, the quotient will be the multiplier. And as the multiplier and multiplicand may be taken reciprocally for each other, *if the product of a multiplication be divided by one of its factors, the quotient will be the other factor.*

For example, having found in article (50,) that 2864 multiplied by 6, gives 17184 for product, we divide 17184 by 2864 and the quotient is the multiplier, 6.

In like manner, as the quotient in division expresses how many times the dividend contains the divisor; if the divisor be taken as many times as there are units in the quotient, that is, if the divisor be multiplied by the quotient, the product will be the dividend, if the division was performed without a remainder. But if there is a remainder it must be added to the product.

For example, having found as in article (63,) that 189492 divided by 375, gives 505 for quotient and 117 for remainder; if we multiply 375 by 505 and add 117 to the product, the dividend, 189492, will be reproduced.

Thus Multiplication and Division reciprocally prove each other.

But these operations may be verified by a more ready method, which will now be explained. The preceding reflections, however, should not be neglected, as they will be useful on many occasions.

PROOF BY 9.

76. Let us suppose, that having multiplied 65498 by 454 and found 29736092 for the product, we wish to know if this product is exact.

The figures of the multiplicand 6, 5, 4, 9, 8, are added together in the same manner as simple units, and their sum divided by 9. After division, 5 is found for remainder.

In like manner, the figures 4, 5, 4, of the multiplier, are added together, and their sum divided by 9. After division, 4 is found for remainder.

The remainders, 5 of the multiplicand, and 4 of the multiplier, are then multiplied together, and the product 20 divided by 9. After this division, 2 is found for remainder.

If the product to be verified is exact, on adding its figures 2, 9, 7, 3, 6, 0, 9, 2, and dividing their sum by 9, 2 will be found for remainder; which is the case.

As Division is an abridged method of performing Subtraction, this rule may be considered to be founded on this principle, that, *to have the remainder after subtracting all the nines contained in a given number, we need only seek the remainder after suppressing all the nines contained in the sum of its figures added as simple units.*

Thus, if all the nines are subtracted from a number expressed by a single figure followed by many ciphers, the remainder will be expressed by the same figure. If from 400, or 5000, or 60000, &c. all the nines are subtracted, the remainder will be 4, or 5, or 6, &c.

Then the remainder, which results from suppressing all the nines in such a number as 65498 (which is the same number as 60000, plus 5000, plus 400, plus 90, plus 8), will be the same as that given by 6, plus 5, plus 4, plus 9, plus 8; that is, the same as that given by adding all these figures as simple units.

Let us now explain the application of this proof to multiplication.

Since 65498 is composed of a certain number of nines and the remainder 5, and as the multiplier 454 is also composed of a certain number of nines and the remainder 4; in order that the total product may be divisible by 9, the product of 4 by 5, or 20 only is requisite; or, in casting out the nines, only 2 is wanting to render the total product divisible by 9. Then there should be the same quantity remaining from the product, as in the product of the two remainders after casting out the nines they contain.

The same proof may also be made by the figure 3.

Division may be easily proved by observing what has been said in article (70.) After subtracting from the dividend the remainder given by division, the result may be considered as a product, of which the divisor and quotient are the factors; consequently, proof by 9 may be applied in the same manner as in the preceding case.

USES OF THE PRECEDING RULE.

77. **Division is not only useful** for finding how often one number contains another, but also for separating a number into equal parts. The half, third, fourth, fifth, twentieth, thirtieth, &c. part of a number may be taken by dividing the number by 2, 3, 4, 5, 20, 30, &c. or separating the number into 2, 3, 4, 5, 20, 30, &c. equal parts, to take one of those parts.

Division is also used for changing units of a certain kind into those of a higher denomination retaining the same value; for example, to reduce pence to shillings, and shillings to pounds. It may be remarked, that when pence are to be converted into shillings, the pence must be divided by the number which expresses the number of pence in a shilling, and the quotient will be the shillings in the given pence. Thus, to reduce 5864 pence to shillings, we divide 5864 by 12, and 488, the quotient, is the number of shillings in 5864 pence. The remainder 8 is pence. To reduce 488 shillings to pounds, we divide 488 by 20, because 20 shillings make 1 pound, and the quotient 24 is pounds; and the remainder 8 is shillings. So the total value of 5864 pence, expressed in pounds, shillings, and pence, is £24 8*s.* 8*d.*

The principle explained in the preceding example may be summed up in this general rule. *Divide the lower denomination by the number which expresses an unit of the next higher, and the quotient will be of the higher denomination. Divide this quotient by the number expressing an unit of the next higher denomination; and thus continue to the last. The last quotient and the several remainders will be equivalent to the number to be reduced.*

EXAMPLES FOR PRACTICE.

ENGLISH MONEY.

In 46716 pence how many pounds?
Ans. £486 13*s.* 8*d.*

2. In 9752 pence, how many pounds?
Ans. £40 12*s*, 8*d.*

3. In 48960 farthings, how many pence, three-pences, and six-pences? Ans. 12240 pence, 4080 three-pences, and 2040 six-pences.

FEDERAL MONEY.

4. In 54000 mills, how many cents? Ans. 5400.

5. In 340000 cents, how many dollars? Ans. 3400.

6. In 9500000 mills, how many cents, dimes, and dollars? Ans. 950000 cents, 95000 dimes, and 9500 dollars.

The following examples in the reduction of weights, measures, &c. furnish a proof to corresponding examples in article (57.) This proof is founded on the general principle, that multiplication and division reciprocally prove each other.

TROY WEIGHT.

7. In 13853 grains, how many pounds, &c.?

13853 must be divided by 24, the quotient of that division by 20, and the quotient thence arising by 12.—The operation is as follows.

```
          gr.
  13853 | 24     dwt.
  120   | ----- | 20    oz.
  ----- | 577   | ---- | 12
    185   40    | 28   | --
    168   ----    24   |  2 lbs.
  -----   177     ----
    173   160     4 oz.
    168   ----
  -----     17 dwt.
      5 gr.
```

The result, or answer, is 2lb 4oz. 17dwt. 5gr.

8. How many pounds in 106317 grains?
Ans. 18℔ 5oz. 9dwt. 21grs.

9. If a silver cup weighing 9oz, 4dwt. 16grs. be sold for $13, 29cts. 6m.; what is it per grain?
Ans. 3 mills.

AVOIRDUPOIS WEIGHT.

10. In 1136260 drams, how many hundred weight?
Ans. 39cwt. 2qr. 14lb. 8oz. 4drs.

11. In 286720 ounces, how many tons? Ans. 8.

12. In 31920 pounds of sugar, how many hogsheads are there, each hogshead weighing 9½ cwt.? Ans. 30.

LONG MEASURE.

13. In 320 rods, how many miles? Ans. 1.

14. The distance from Boston to Salem is 31680 yards, how many miles is it? Ans. 18.

15. In 4[illegible]05728000 barley corns, how many degrees?
Ans. 360.

CLOTH MEASURE.

16. In 1344 nails, how many yards? Ans. 84.

17. In 1566 yards of Holland, how many pieces, each containing 36 ells Flemish? Ans. 58.

18. If 13 ells Fl. 3 qrs. cost $70,56, what was it per nail? Ans. 42 cents.

TIME.

19. In 350400 hours, how many years? Ans. 40.

20. In 240 days, how many weeks?
Ans. 34 weeks and 2 days.

21. Since the birth of our Saviour there have been 955540800 minutes; what year is it? Ans. 1818.

LAND OR SQUARE MEASURE.

22. In 7040 rods, how many acres? Ans. 44.

23. In 7938 perches, how many acres? A. R. per.
Ans. 49 2 18.

24. In 136 roods, how many acres? Ans. 34.

WINE MEASURE.

25. In 1260 gallons of wine, how many tuns? Ans. 5.
26. In 5040 pints of wine, how many pipes? Ans. 5.
27. In 6048 gills, how many hogsheads? Ans. 3.

APOTHECARIES' WEIGHT.

28. In 2998 drams, how many pounds?
Ans. 31℔ 2℥ 6ʒ.
29. In 55799 grains, how many pounds?
Ans. 9℔ 8℥ 1ʒ 2sc. 19grs.
30. In 80640 grains, how many pounds? Ans. 14℔.

DRY MEASURE.

31. In 960 bushels of coal, how many chaldrons?
Ans. 30.
32. In 1536 pints, how many bushels? Ans. 24.
33. If 5bush. 2pks. 3qts. of salt cost $5,87, what was that per quart? Ans. 3 cents.

SOLID MEASURE.

34. In 829440 cubic inches, how many tons of round timber? Ans. 12.
35. In 1036800 cubic inches, how many tons of hewn timber? Ans. 12.
36. In 221184 solid inches, how many cords of wood
Ans. 1
37. A farm of 375 acres is let for $1125; how much does it pay per acre? Ans. 3 dollars.
38. A certain number of men were concerned in the payment of 18950 dollars, and each man paid 25 dollars, what was the number of men? Ans. 758.
39. What number must be multiplied by 13, that the product may be 871? Ans. 67.
40. Supposing a man's income to be 2555 dollars a year; how much is that per day, there being 365 days in a year? Ans. 7 dollars.
41. A brigade of horse consisting of 384 men, is to be formed into a column having 32 men in front; how many ranks will there be? Ans. 12.

42. A prize of 48726 dollars is owned by 270 men; what is each man's share? Ans. 180$\frac{21}{45}$ dollars.

43. How many times is 8 contained in 6,848? Ans. ,856.

44. How many times 3 is 12,873? Ans. 4,291.

45. 86,436 is how many times 9? Ans. 9,644.

46. 1,740 is how many times 6? Ans. ,290.

47. 18,345 is how many times 5? Ans. 3,669.

48. 3,102 is how many times 94? Ans. ,33.

49. A field of 27 acres produced 675 bushels of wheat; what was that per acre? Ans. 25 bushels.

50. If 7412 eggs be packed in 34 casks, how many were in each cask? Ans. 218.

51. An army of 15000 men plundered a city and took 2825000 dollars; what was each man's share? Ans. 175 dollars.

52. If an estate of 36582 dollars is divided among 13 sons, how much does each son receive? Ans. $2814.

53. I would plant 2072 trees in 14 rows, twenty-five feet asunder; how long must the grove be? Ans. 3675 feet.

54. Sixty men at a feast, which lasted 3 days, spent 240 dollars per day; how much did each man spend per day, and how much in the whole? Ans. $4 per day, $12 in all.

55. Divide 151200 lbs. of beef equally among an army consisting of 27 regiments, each regiment of 7 companies, and each company 100 men; and what allowance will each man receive? Ans. 8lbs.

REDUCTION OF CURRENCY.

The coins of the United States, as established by the law regulating the mint, are three of gold, six of silver, and two of copper. The gold coins are the Eagle, the half-Eagle, and quarter Eagle. For gold, the standard is eleven parts fine, and one alloy. The alloy is a mixture of silver and copper. In the Eagle, the weight of fine gold is 247,5 grains; and of standard gold, 270 grains. The silver coins are dollars, half-dollars, quarter-dollars, double dimes, dimes, and half-dimes. The standard for silver coins is 1485 parts fine, to 179 parts alloy. The alloy is entirely copper. The weight of fine silver in the dollar is 371,25 grains; and of standard silver, 416 grains. The weight of the cent is 208 grains. All coins inferior to the eagle, and dollar, contain a quantity of pure metal and alloy proportional to their denominations. The relative value of gold to silver is established by law in the proportion of 15 to 1. When gold or silver is finer or coarser than the standard, the variation is estimated by *carats* and grains of a carat in gold; and by penny weights in silver. The carat is no fixed quantity, but one twenty-fourth part of any quantity. The carat is divided by goldsmiths and at the mint into four equal parts called *grains* of a carat.

As the divisions in Federal money, are decimal; all rules for the treatment of decimals are applicable to them. But as the currencies of the several States have different values compared to the Federal coin, some rules are necessary for reducing them to a par with each other. For example, the value of the dollar in New-England and Virginia is 6 shillings; in New-York and North-Carolina it is 8 shillings; in New-Jersey, Pennsylvania, Delaware and Maryland, it is 7 shillings 6 pence; in South-Carolina and Georgia it is 4 shillings 8 pence; the value of the shilling varies accordingly. The Spanish and American dollar is 4 shillings 6 pence sterling. Now if it be required to change New-England and Virginia currency to Federal Money; since the dollar is six-twentieths, or three-tenths of a pound, it is evident, that if three ciphers are annexed to the pounds, and the sum divided by 3, the quotient will be dollars.

The following rules are founded on the consideration of the relative values of dollars, cents, &c. to pounds, shillings, &c.

TO REDUCE NEW-ENGLAND AND VIRGINIA CURRENCY TO FEDERAL MONEY.

If the sum consist of pounds only, annex to it three ciphers and divide by 3; the quotient will be the answer in cents. When pounds and shillings are given, annex to the pounds half the number of shillings when they are an even number, and two ciphers; but if the number be odd, annex half the even number, then 5 for the odd number, *the one cipher; divide this sum by 3 and the quotient will be cents. When pounds, shillings, pence, and farthings are in the given sum, annex for the shillings as already directed, and to these add the farthings contained in the pence and farthings, observing to augment their number by unity when they exceed 12, and by 2 when they exceed 36; and divide as before.*

EXAMPLES.

1. Reduce £34 16s. 4d. 2fa. to Federal money.

The operation is as follows.

34000 pounds with 3 ciphers annexed.
8 half the even number of shillings.

34800 half the shillings and two ciphers.
19 the farthings in the pence and farthings augmented by unity.

34819 | 3 divisor.
| 11606⅓ cents quotient.

2. Reduce £471 to Federal money. Ans. $1570.
3. Reduce £37 to dollars and cents. Ans. $123,33⅓.
4. Reduce £64 16s. to dollars and cents. Ans. $216,00cts.
5. Reduce £1 17s. to dollars. Ans. $6,16⅔.
6. Reduce £591 11s. 9d. 2qrs. to dollars. $1971,96⅔ cts.

TO REDUCE FEDERAL MONEY TO NEW-ENGLAND AND VIRGINIA CURRENCY.

When the sum is in dollars only, multiply by 3, and double the product of the first figure for shillings; the remaining part of the product will be pounds. When cents are in the given sum, multiply the whole by 3, and cut off three figures to the right hand as a remainder; or if cents and mills are in the sum, cut off four figures for a remainder; multiply the remainder by 20, and cut off as before. Proceed in the same manner with the several parts of the pound; and the parts on the left are the answer in its several denominations.

EXAMPLES.

1. Reduce 473 dollars to New-England currency.
The following is the method of operation.

473
3

£141 9
2

£141 18 shillings.

2. Reduce $429 21cts. 5mills to N. Eng. currency.
The operation is conducted thus.

429215
3

£128,7645
20

s. 15,2900
12

d. 3,4800
4

qr. 1,9200
Ans. £128 15s. 3d. 1qr.

3. How many pounds, &c. in 579 dollars?
Ans. £173 14s.

4. Reduce $609 88cts. 3mills to New-England currency.
Ans. £182 19s. 3d. 2qrs.

To change New-York and North Carolina currency to Federal money, the dollar being 8 *shillings.*

Prepare the sum as for New-England currency, and divide by 4; and the quotient is the answer in cents.

EXAMPLES.

1. Change £461 to Federal money.

461000 | 4

Ans. $1152,50cts.

2. Change £419 10s. 8d. 2qrs. to Federal money.
Ans. $1048,83¾ cents.

To change Federal money to New-York and North Carolina currency.

As the value of the dollar is equal to four tenths of the pound, proceed in the same manner as for New-England currency, only using 4 as a multiplier, instead of 3.

EXAMPLES.

1. Change $1048 83¾cts. to New-York currency.

1048,83¾
4

£419,535
20

s. 10,700
12

d. 8,400
4

qr. 1,600

Ans. £419 10s. 8d. 1qr.

2. Change $1684 to New-York and North Carolina currency. Ans. £673 12s.

To change New-Jersey, Pennsylvania, Delaware, and Maryland currency to Federal money, the dollar being 7s. 6d.

As the value of a dollar is three-eighths of a pound, when only pounds are in the given sum, multiply it by 8 and divide by 3 for dollars. If there are shillings, &c. increase the sum in pence by one-ninth of the whole sum for cents.

EXAMPLES.

1. Change £480 19s. 9d. to Federal money.

£480 19s. 9d.
20

9619
12

115437(9
12826⅓ The quotient after dividing by 9, or one ninth of the whole sum.

128263⅓

2. Change £471 to Federal money. Ans. $1256.

To change Federal money to New-Jersey, Pennsylvania, Delaware, and Maryland currency.

When the sum is in dollars, multiply it by 3 and divide by 8 for pounds. If there are dollars and cents, multiply the sum by 90, and the product is pence, after rejecting two figures on the right.

EXAMPLES.

1. Change $1282 63⅓cts. to Pennsylvania currency.

128263⅓
90

115437,00 | 12

9619 9d | 20

£480 19s. 9d. Ans.

2. Change $1256 to Pennsylvania currency.
Ans. £471.

To change Canada and Nova Scotia currency to Federal money, the dollar being 5 shillings.

As the value of a dollar is equal to one-fourth of a pound, when the sum is in pounds, multiply it by 4, for dollars. When there are shillings, &c. reduce the given sum to pence, annex two ciphers and divide by 60, for cents.

EXAMPLES.

1. Change £528 12s. 6d. Canada currency to Federal money.

20

10572
12

1268700 | 0 | 60

211450 Ans. $2114,50cts.

2. Change £36 Canada currency to Federal money.
Ans. $144.

To change Federal money to Canada and Nova Scotia currency.

Divide the sum in dollars by 4, for pounds. If dollars and cents are in the given sum, multiply it by 60, the product, after rejecting two figures on the right, will be pence.

EXAMPLES.

1. Change $144 to Canada currency. Ans. £36.
2. Change $2114,50cts. to Nova Scotia currency. Ans. £528 12s. 6d.

To change pounds sterling to Federal money.

Reduce the pounds to pence, to the sum annex two ciphers, and divide it by 54; the answer will be cents.

EXAMPLES.

1. Reduce £25 sterling to Federal money. Ans. $111,11cts. and rem.
2. Reduce £437 18s. sterling to Federal money. Ans. $1496,22cts.

General rule for changing the currency of each State to Federal money.

Divide the given sum, reduced to shillings, six-pences, or pence, by the number of shillings, sixpences or pence, in a dollar, as it passes in each State.

EXAMPLES.

1. Reduce £63 15s. New-England, or Virginia currency, to Federal money, the dollar being 6s. Ans. $212,50cts.
2. Reduce £112 15s. New-York currency to Federal money. Ans. $282,00.

FRACTIONS.

78. *Fractions, considered arithmetically, are numbers, which represent quantities smaller than unity.*

A correct idea of Fractions may be formed by considering the quantity, which was first taken for unity, to be divided into a certain number of parts smaller than that unity; for instance, the unit called a dollar, to be divided into an hundred equal parts called cents.

One or more of these parts, or subdivisions of the principal unit, form what is called a fraction of the unit.—The same name is also given to the numbers, which represent these parts.

79. A fraction can be expressed in numbers by two methods, both of which are in use. The first consists in representing as whole numbers, the parts of the unit which contain the quantity considered; a particular name being then given to those parts. Thus, to express 7 parts of the 20 into which we consider the pound to be divided, we employ the figure 7, but call it 7 shillings, and write 7s. This method is employed in expressing the parts of unity instead of complex numbers, of which we shall speak hereafter.

80. But as a particular sign is required for every division of the unit, we avoid a multiplicity of signs, by expressing the fraction by two numbers placed one below the other and separated by a line. Thus, to express the 7 parts in question, we write $\frac{7}{20}$; or, in general, that number is first written which expresses how often the quantity considered contains parts of the unit; and below this, the number, which expresses the number of parts into which the unit is supposed to be divided.

81. In reading a fraction, the upper number, which is called the *numerator*, is pronounced first; then the lower number, which is called the *denominator*. But to the name of the last we add the termination *eths*, or *ths*.—Thus, in reading $\frac{7}{20}$, we call it *seven twentieths*. In reading the fraction $\frac{4}{5}$, we say *four fifths*. By the expression four fifths, we mean four parts of the five, into which the unit is supposed to be divided.

82. The numerator, therefore, expresses the parts of the unit, which are contained in the given quantity; and the denominator teaches the value of those parts, by showing how many of them compose the unit. This part of the fraction is called the denominator, because it gives a name to the whole fraction, and causes in the fractions $\frac{4}{5}$ and $\frac{4}{7}$, for instance, the parts of the first to be called *fifths*, and the parts of the second to be called *sevenths*.

83. The numerator and the denominator are also called, by a common name, *the two terms of the fraction*.

WHOLE NUMBERS CONSIDERED UNDER THE FORM OF A FRACTION.

84. The operations, which are performed upon fractions, often lead to fractional results, in which the numerator is equal to, or greater than the denominator; for example, to such results as $\frac{6}{6}$, $\frac{27}{5}$, &c.

Such fractional expressions as are equivalent to a whole number, or to whole numbers joined to fractions, are called *improper fractions.* Hence $\frac{6}{6}$, $\frac{8}{6}$, and $\frac{12}{6}$ are *improper* fractions, because they answer respectively to 1, $1\frac{2}{6}$, and 2; while $\frac{4}{6}$ is a *proper* fraction, being below unity.

85. *To extract the whole number or numbers found in an improper fraction, divide the numerator by the denominator and the quotient will be the answer sought; and if there be a remainder, set it above the denominator, and let it follow the quotient as a separate expression.* Thus, $\frac{4}{4}$ is equal to 1; and $\frac{5}{4}$ is equal to $1\frac{1}{4}$. This last is called a *mixed* number, because formed of an entire number and a fraction.

86. The multiplications and divisions, which are called for in cases of whole numbers accompanied with fractions, requires, that the whole value should be expressed in the form of an improper fraction.

This is accomplished by multiplying the whole number by the denominator of the fraction, which we wish to form, and making the result the numerator of this denominator. For example, when 5 is to be brought into fifths, we multiply 5 by 5, and place the result, which is 25, above the multiplier, and $\frac{25}{5}$ will be the fraction required. If a mixed number be in question, as $5\frac{2}{7}$; then to 35 obtained in the manner just directed, add 2, taken from the numerator of the accompanying fraction, and we have $\frac{37}{7}$ for the fraction sought.

EXAMPLES FOR PRACTICE UNDER ARTICLES 85 AND 86.

1. Reduce $\frac{17}{5}$ to its equivalent whole or mixed number. Ans. $3\frac{2}{5}$.

2. Reduce $\frac{245}{9}$ to a whole or mixed number. Ans. $27\frac{2}{9}$.

3. Reduce $\frac{8229}{16}$ to a whole or mixed number. Ans. $514\frac{5}{16}$.

4. Reduce $5\frac{3}{8}$ to an improper fraction. Ans. $5\times8+3=\frac{43}{8}$.

5. Reduce $183\frac{5}{21}$ to an improper fraction. Ans. $\frac{3848}{21}$

6. Reduce $27\frac{2}{9}$ to an improper fraction. Ans. $\frac{245}{9}$.

7. Reduce $514\frac{5}{16}$. to an improper fraction. Ans. [illegible]$\frac{}{16}$.

CHANGES MADE IN THE TERMS OF A FRACTION WITHOUT ALTERING ITS VALUE.

87. It is evident, that the greater number of parts unity is supposed to be divided into, the greater must be the number of those parts to compose a given quantity. Thus divide the dollar into two parts, and take one, you have $\frac{1}{2}$ of a dollar. If you divide it into twice as many, that is, 4 parts, you must take two of them or $\frac{2}{4}=\frac{1}{2}$ to express the same quantity.

88. Then the denominator of a fraction may be doubled, tripled, quadrupled &c. without changing the value of the fraction; if, at the same time, its numerator is doubled, tripled, quadrupled, &c.

Hence, *the value of a fraction is not changed when both of its terms are multiplied by the same number.*

Thus the value of $\frac{3}{4}$ is the same as that of $\frac{6}{8}$; and $\frac{1}{2}$ is the same as $\frac{2}{4}$, $\frac{3}{6}$, $\frac{4}{8}$, &c.

89. It is evident from similar reasoning, that if the unit be divided into only a small number of parts, there will be a less number of parts required to form a given quantity. Consequently, the denominator of a fraction may be rendered 2, 3, 4, &c. times smaller, without changing its value; if, at the same time, the numerator be rendered 2, 3, 4, &c. times smaller. And in general, *the value of a fraction cannot be changed by dividing both of its terms by the same number.*

To perceive more clearly the truth of these two propositions, we need only recollect the definition, which has been given of the numerator and denominator of a fraction. And it may be remarked, that to multiply or divide the two terms of a fraction by the same number, is the same thing as to multiply or divide the fraction; since its value is not changed by these operations, as has already been seen.

The two principles just established form the basis of the following reductions, which are of important use.

REDUCTION OF FRACTIONS TO THE SAME DENOMINATOR.

90. 1st. *To reduce two fractions to the same denominator, multiply the two terms of the first fraction, each by the denominator of the second; and both terms of the second, each by the denominator of the first.*

For example, to reduce the two fractions $\frac{2}{3}$, $\frac{3}{4}$, to the same denominator, we multiply 2, and 3, which are the two terms of the first fraction, each by 4, the denominator of the second, and have $\frac{8}{12}$, which (88) has the same value as $\frac{2}{3}$.

We likewise multiply 3 and 4, the two terms of the second fraction, each by 3, the denominator of the first, and have $\frac{9}{12}$, which is of the same value as $\frac{3}{4}$. The two fractions $\frac{2}{3}$ and $\frac{3}{4}$ are now changed into $\frac{8}{12}$ and $\frac{9}{12}$, which respectively have the same value as the first; and they have the same denominator.

Hence it may be seen, that each of the new fractions will always have the same denominator; because in each operation the new denominator is formed by the multiplication of the two first denominators.

91. 2d. *When more than two fractions are in the given sum, they may be reduced to a common denominator, by multiplying the two terms of each by the product, which results from the multiplication of the denominators of the other fractions.*

For example, to reduce these fractions $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, $\frac{5}{7}$, to the same denominator, we multiply 2 and 3, the two terms of the first fractions, by the product of 4, 5, 7, the denominators of the other fractions. This product is found by saying, 4 times 5 make 20, and 7 times 20 are 140. We then multiply 2 and 3, each by 140, and have $\frac{280}{420}$, which is of the same value as $\frac{2}{3}$ (88).

We multiply, in like manner, 3 and 4, the two terms of the second fraction, each by the product of 3, 5, 7, which gives $\frac{315}{420}$, a fraction, which has the same value as $\frac{3}{4}$.

Passing to the third fraction, we multiply 4 and 5, its two terms, each by 84, the product of the three denominators 3, 4, and 7, and have $\frac{336}{420}$ instead of $\frac{4}{5}$.

At the fourth fraction, we multiply 5 and 7, each by 60, the product of 3, 4, 5, the denominators of the three first fractions, and have $\frac{300}{420}$ instead of $\frac{5}{7}$. The four fractions $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, $\frac{5}{7}$ are now changed into $\frac{280}{420}$, $\frac{315}{420}$, $\frac{336}{420}$, $\frac{300}{420}$, less simple than the first, but by reason of their common denominator more susceptible of the operations of addition and subtraction.

It may be remarked, that the denominator of each new fraction being formed of the product of all the first denominators, must be the same for each fraction.

The rules contained in the two last articles are useful in performing such examples as the following.

EXAMPLES.

1. Reduce $\frac{3}{4}$, $\frac{5}{9}$, and $\frac{7}{12}$, to a common denominator.
Ans. $\frac{27}{36}$, $\frac{20}{36}$, $\frac{21}{36}$.

2. Reduce $\frac{1}{2}$, $\frac{2}{3}$, $\frac{5}{6}$, and $\frac{7}{8}$, to a common denominator.
Ans. $\frac{12}{24}$, $\frac{16}{24}$, $\frac{20}{24}$, $\frac{21}{24}$.

3. Reduce $\frac{2}{3}$, $\frac{4}{9}$, $\frac{3}{7}$, and $\frac{5}{21}$, to a common denominator.
Ans. $\frac{42}{63}$, $\frac{28}{63}$, $\frac{27}{63}$, $\frac{15}{63}$.

4. Reduce $\frac{1}{3}$, $\frac{3}{5}$, $\frac{4}{15}$, and $\frac{5}{9}$, to a common denominator.
Ans. $\frac{15}{45}$, $\frac{27}{45}$, $\frac{12}{45}$, $\frac{25}{45}$.

Although the value of a fraction is not changed when both of its terms are multiplied or divided by the same quantity, it may be observed;

1st. *That if the numerator is multiplied, or the denominator divided by any number, the fraction is multiplied by that number.* For example, if the fraction $\frac{3}{16}$ is to be multiplied by 4, we may either multiply its numerator by 4, which gives $\frac{12}{16}$, or divide its denominator by 4, which gives $\frac{3}{4}$. For the two results are the same; because, by dividing the two terms of the first result by 4, which does not change its value (88), we obtain $\frac{3}{4}$, the second result.

2d. *If the numerator be divided, or the denominator multiplied by any number, the fraction is divided by that number.* For example, to divide the fraction $\frac{8}{12}$, we can, as in the last case, either multiply its denominator by 4, which gives $\frac{8}{48}$, or divide its numerator by 4, which gives $\frac{2}{12}$. For the two results $\frac{8}{48}$ and $\frac{2}{12}$ are the same; because, if $\frac{8}{48}$, the first result, is divided by 4, which does not change its value, (89) the first result $\frac{2}{12}$, will be obtained.

REDUCTION OF FRACTIONS TO THEIR MOST SIMPLE EXPRESSION.

92. A fraction is simplified by expressing, in a smaller number of figures, the value of its two terms. This simplification takes place when the numerator and denominator are divisible by the same number. Their value is not changed by this operation (89), which is conducted in the following manner.

93. *Divide the numerator and denominator, each by 2, and repeat the division till it cannot be performed without a remainder. Then divide both the terms by 3, until they cannot be divided without a remainder. Continue the same operation by dividing both terms of the fraction successively by the numbers*, 5, 7, 11, 13, 17, &c.; that is, by numbers which have no divisor but themselves. These numbers are called *prime numbers.*

The only difficulty in these operations is to know when the terms of the fraction are divisible by 2, 3, 5, &c. But the following observations may be useful in removing it.

94. Every number terminated by a figure which expresses an even number of units, will be divisible by 2. Any number, the sum of whose figures added together as simple units, make 3 or a multiple of 3, that is, an exact number of times 3, will be divisible by 3. For example, 54231 is divisible by 3, because the sum of its figures 5, 4, 2, 3, 1, is 15, which is 5 times 3. The same property belongs to the number 9, or to a multiple of 9. The reason of this rule is founded on a principle similar to that demonstrated in article (75).

Every number, terminated by 5, or a cipher, is divisible by 5.

Similar rules might be found for 7, and the numbers following. But as they require a long operation, it is better to use division.

It is required to reduce the fraction $\frac{2016}{5796}$. We divide both the terms by 2, because the last figure in each term is an even number, and have $\frac{1008}{2898}$. We divide again by 2 and have $\frac{504}{1449}$. We next divide by 3, according to the rule given above, and have $\frac{168}{483}$. We divide again by 3 and have $\frac{56}{161}$. We now attempt to divide by 7; the division succeeds, and we have $\frac{8}{23}$.

The reason why we have directed, that division should be attempted only by the prime numbers 2, 3, 5, 7, &c., is, that having exhausted the division by 2, for instance, it would be useless to attempt it by 4, because 4 is a multiple of 2.

95. But the readiest method for reducing a fraction to its most simple expression, is to divide both its terms by their greatest common divisor. The rule for finding the greatest common divisor is as follows.

Divide the greater of the two terms by the less, and if there is no remainder, the less term is the greatest common divisor.

But if there be a remainder, divide the less term by it; if the division is exact, the first remainder is the greatest common divisor.

If the second division leave a remainder, divide the first remainder by the second; and continue thus to divide the preceding remainder by the last, until there is an exact division. Then the last divisor will be the greatest common divisor of the two terms of the fraction.

When the last divisor is unity, it is a proof that the fraction cannot be reduced.

For illustrating this rule take the fraction $\frac{3760}{9024}$. We divide 9024 by 3760 and have 2 for quotient and 1504 for remainder; then 3760 by 1504 and 2 is the quotient and 752 the remainder The first remainder 1504 is then divided by the second remainder 752; the division is exact. We conclude therefore, that 752, is the greatest common divisor of the fraction $\frac{3760}{9024}$. On dividing both terms by it, the operation gives $\frac{5}{12}$, the simplest expression of the fraction.

As 752 divides 1504, it ought also to divide 3760, which is composed of twice 1504 added to 752. And as 9024 is composed of twice 3760 added to 1504, it ought to be divisible by 752.

It is also plain, that 752 is the greatest common divisor, which 3760 and 9024 can have; because there can be no divisor common to 9024 and 3760, which is not common to 3760 and 1504; and to these last there can be no common divisor, which is not common to 1504 and 752. And to the last there can be no greater divisor than 752.

EXAMPLES FOR PRACTICE.

1. Reduce $\frac{144}{140}$ to its simplest expression. Ans. $\frac{3}{5}$.
2. Reduce $\frac{48}{272}$ to its simplest expression. Ans. $\frac{3}{17}$.
3. Reduce $\frac{192}{576}$ to its simplest expression. Ans. $\frac{1}{3}$.
4. Reduce $\frac{825}{960}$ to its simplest expression. Ans. $\frac{55}{64}$.
5. Reduce $\frac{252}{364}$ to its simplest expression. Ans. $\frac{9}{13}$.
6. Reduce $\frac{5184}{6912}$ to its simplest expression. Ans. $\frac{3}{4}$.
7. Reduce $\frac{1344}{1536}$ to its simplest expression. Ans. $\frac{7}{8}$.
8. Reduce $\frac{6896800}{36700160}$ as much as possible.

Ans. $\frac{43105}{229376}$.

DIFFERENT WAYS OF CONSIDERING A FRACTION AND THE CONSEQUENCES WHICH FOLLOW.

96. The idea we have given of a fraction is, that the denominator shows the number of parts of which the unit is composed; and the numerator, the number of parts in the quantity, which the fraction expresses.

97. The numerator of a fraction may also be considered as a dividend, and the denominator as a divisor. Hence the reason why the remainders of division were put under the form given them in article (60).

98. From these considerations it follows, 1st. that a whole number may be put under the form of a fraction, by making the whole number the numerator, and giving to it unity for a denominator. Thus 8 or $\frac{8}{1}$ are the same thing; and so are 5 or $\frac{5}{1}$, 10 or $\frac{10}{1}$, &c.

99. 2d. Any fraction may be changed into decimals by considering the numerator as the remainder of a division in which the denominator was divisor, and operating according to the direction given in (page 59)—observing to put a cipher in place of units in the quotient. Thus, we find that 0,6 is the decimal expression for $\frac{3}{5}$, 0,555 for $\frac{5}{9}$, 0,84 for $\frac{21}{25}$; and so on.

On this principle is founded a rule for reducing complex numbers to decimals. For example, if it is required to reduce 3 fathoms, 5 feet, 8 inches, 7 lines, to the decimal of the fathom expressed in half-lines; we observe, that the fathom contains 864 lines, and consequently

1728 half-lines; therefore, in order to express the decimal value in half-lines, we must continue the operation to the fourth place of decimals. The 5ft. 8in. 7l. are now reduced to lines; and we have 823 lines or $\frac{823}{864}$ of a fathom. After reducing this fraction to decimals in the manner directed, we have 0,9525; and consequently 3,9525fa. for the proposed number. The following rule, however, is in more common use.

" *To reduce a number from a lower denomination to the decimal of a higher, we first change it, or suppose it to be changed, into a fraction having* 10, *or some multiple of* 10, *for its denominator, and divide the numerator by as many as make one of this higher denomination, and the quotient is the required decimal; which, together with the whole number of this denomination, may again be converted into a fraction, having* 10 *or a multiple of* 10 *for its denominator, and thus by division be reduced to a still higher name, and so on.*

EXAMPLES FOR PRACTICE.

1. Reduce 19s. 3d. 3qr. to the decimal of a pound.

The operation is as follows.

4	3,00
12	3,75000
20	19,312500
	0,965625

In this example, 3, the number of farthings, is considered as $\frac{300}{100}$qr. which are reduced to hundredths of a penny by dividing by 4, the figure on the left, and the quotient 75 is placed as a decimal on the right of the pence; we then take this sum, considered as $\frac{375}{100}$d. or $\frac{3750}{1000}$d. that is, annex as many ciphers as are necessary, and divide it by 12, which brings it into decimals of a shilling. Lastly, the shillings, and parts of a shilling, 19,3125s. considered as $\frac{19312500}{1000000}$s. are reduced to decimals of a pound by dividing by 20, which gives the result above found.

2. Reduce 17pls. 1ft. 6in. to the decimal of a mile.

12	6
16,5	1,5
320	17,09
	0,00531531 and rem.

3. Reduce 9s. to the decimal of a pound. Ans. ,45.

4. Reduce 19s. 5d. 2qr. to the decimal of a pound. Ans. ,972916.

5. Reduce 10oz. 18dwt. 16gr. to the decimal of a pound Troy. Ans. ,911111, &c.

6. Reduce 2qrs. 14℔. to the decimal of a cwt. Ans. ,625, &c.

7. Reduce 17yd. 1ft. 6in. to the decimal of a mile. Ans. ,00994318, &c.

8. Reduce 3qrs. 2nls. to the decimal of a yard. Ans. ,875.

9. Reduce 1gal. of wine to the decimal of a hogshead. Ans. ,015873.

10. Reduce 3bus. 1pk. to the decimal of a quarter. Ans. ,40625.

11. Reduce 10w. 2d. to the decimal of a year. Ans. ,1972602, &c.

OPERATIONS OF ARITHMETIC UPON FRACTIONS.

100. The same operations are performed upon fractions as upon whole numbers. Addition and Subtraction often require a preparatory operation; Multiplication and Division require very little preparation.

ADDITION OF FRACTIONS.

101. *If the fractions have the same denominator, add all the numerators, and affix to their sum the common denominator.* Thus, to add $\frac{2}{7}$, $\frac{3}{7}$, and $\frac{5}{7}$, we add together 2, 3, and 5, the numerators, and have $\frac{10}{7}$, which is reduced to $1\frac{3}{7}$, (65.)

102. *If the fractions have not the same denominator, they must be reduced to a common denominator by following the rule given in articles* (90) and (91.) *Then add*

the new fractions as in the preceding manner. Thus, if it is proposed to add $\frac{3}{4}$, $\frac{2}{3}$, $\frac{4}{5}$, these fractions are changed into $\frac{45}{60}$, $\frac{40}{60}$, $\frac{48}{60}$, of which the sum is $\frac{133}{60}$, which is reduced to $2\frac{13}{60}$, (85.)

EXAMPLES FOR PRACTICE.

1. Add $\frac{2}{3}$, $\frac{3}{4}$, $\frac{5}{6}$, $\frac{7}{8}$, and $\frac{11}{12}$ together. Ans. $4\frac{1}{24}$.
2. Add $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{7}$, and $\frac{1}{8}$ together. Ans. $1\frac{61}{280}$.
3. Add $\frac{4}{5}$, $\frac{5}{6}$, $\frac{6}{7}$, $\frac{3}{8}$, and $\frac{8}{15}$ together. Ans. $3\frac{67}{168}$.
4. Add $5\frac{2}{3}$, $6\frac{7}{8}$, and $4\frac{1}{2}$ together. Ans. $17\frac{1}{24}$.
5. Add $1\frac{1}{3}$, $\frac{4}{15}$, and $9\frac{3}{20}$ together. Ans. $11\frac{1}{60}$.
6. Add $1\frac{9}{10}$, $6\frac{7}{8}$, $\frac{2}{6}$, and $7\frac{1}{2}$ together. Ans. $16\frac{73}{120}$.

SUBTRACTION OF FRACTIONS.

103. *If the two proposed fractions have the same denominator, the numerator of one is subtracted from the numerator of the other, and to the remainder the common denominator is affixed.* If it is required to subtract $\frac{5}{9}$ from $\frac{8}{9}$, the remainder will be $\frac{3}{9}$, which is reduced to $\frac{1}{3}$, (93.)

104. If we wish to subtract $4\frac{7}{8}$ from $9\frac{5}{8}$, as $\frac{7}{8}$ cannot be taken from $\frac{5}{8}$, we borrow 1 from 9, which reduced to eighths and added to $\frac{5}{8}$, make $\frac{13}{8}$, from which taking $\frac{7}{8}$, there will remain $\frac{6}{8}$. Then taking 4 from the 8 which remains after 1 was borrowed, we have in all $4\frac{6}{8}$, or $4\frac{3}{4}$.

105. *If the fractions have not the same denominator, they must be reduced to a common denominator,* (90) and (91.) *Subtraction is then performed as in the preceding case.* Thus, to take $\frac{2}{3}$ from $\frac{3}{4}$, these fractions are changed to $\frac{8}{12}$ and $\frac{9}{12}$; then subtracting 8 from 9 there remains $\frac{1}{12}$.

EXAMPLES FOR PRACTICE.

From	$\frac{5}{7}$	$\frac{11}{12}$	$\frac{15}{16}$	$\frac{17}{35}$	$\frac{105}{209}$	3	6
Take	$\frac{2}{7}$	$\frac{5}{12}$	$\frac{3}{16}$	$\frac{13}{35}$	$\frac{9}{209}$	$0\frac{3}{16}$	$\frac{7}{8}$
Rem.	$\frac{3}{7}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{4}{35}$	$\frac{6}{209}$	$2\frac{12}{16}$	$5\frac{1}{8}$

From	$\frac{5}{8}$	$\frac{11}{12}$	$\frac{5}{8}$	$\frac{8}{15}$	$\frac{209}{216}$	$13\frac{1}{9}$	$10\frac{3}{10}$	$12\frac{5}{12}$
Take	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{9}{20}$	$\frac{7}{144}$	$8\frac{14}{27}$	$1\frac{7}{12}$	$6\frac{1}{2}$
Rem.	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{10}$	$\frac{1}{12}$	$\frac{397}{432}$	$4\frac{16}{27}$	$8\frac{43}{60}$	$5\frac{11}{12}$

As it has been seen (88) and (89), that the value of a fraction is not changed by multiplying or dividing both of its terms by the same number; we are naturally led to examine what effect is produced when the same quantity is added to, or subtracted from, both terms of the fraction. Let $\frac{2}{7}$ be a given fraction; by adding 3 to each of its terms, the fraction becomes $\frac{5}{10}$, or $\frac{1}{2}$. But this last fraction is greater than the first, as may be proved by observing, that the fractions $\frac{2}{7}$ and $\frac{1}{2}$ are equivalent to $\frac{4}{14}$ and $\frac{7}{14}$; for, *if the numerator or denominator of a fraction is multiplied or divided by the same number, the value of the fraction is not changed. But,* $\frac{7}{14}$ *is greater than* $\frac{4}{14}$, *the fraction* $\frac{1}{2}$ *is then greater than the fraction* $\frac{2}{7}$. For, *a fraction is augmented or diminished, by augmenting or diminishing its denominator, the numerator remaining the same.* Hence it is plain, *that by adding the same number,* 3, *to both terms of the fraction* $\frac{2}{7}$, *(equivalent to* $\frac{4}{14}$*) it is augmented and becomes* $\frac{1}{2}$ *or* $\frac{7}{14}$.

This is a general property. For, as the value of a fraction is not changed, by adding the numerator and denominator, each to itself; both terms of the fraction may be doubled without altering its value. But, the denominator being always greater than the numerator, in order that the value of the fraction may remain the same, *a greater number must be added to the denominator than to the numerator.* Consequently, by adding the same number to both terms, we add too little to the denominator; the denominator then is too small, and the fraction too large, according to what has been said above. Hence it is plain, that *a fraction is augmented by adding the same number to both its terms.* And reciprocally, *a fraction is diminished when the same number is subtracted from both its terms.* The multiplication and division of one fraction by another is deduced from the preceding principles.

MULTIPLICATION OF FRACTIONS.

106. *To multiply a fraction by a fraction, the numerator of one must be multiplied by the numerator of the other, and the denominator by the denominator.* For example, to multiply $\frac{2}{3}$ by $\frac{4}{5}$, we multiply 2 by 4, which gives 8 for the numerator; multiplying, in like manner, 3 by 5, we have 15 for the denominator, and consequently $\frac{8}{15}$ for the product.

The reason for this rule will be more plain if we recollect, that in the multiplication of one number by another, the multiplicand is taken as many times as there are units in the multiplier. Then, to multiply $\frac{2}{3}$ by $\frac{4}{5}$, is to take 4 times the fifth part of $\frac{2}{3}$. But in multiplying the denominator 3 by 5, we change the thirds to fifteenths, that is, into parts five times smaller. And in multiplying the numerator 2 by 4, we take these new parts four times; hence we take four times the fifth part of $\frac{2}{3}$. Therefore $\frac{2}{3}$ is multiplied by $\frac{4}{5}$.

107. If whole numbers are to be multiplied by a fraction, or a fraction by whole numbers, the whole numbers are written as a fraction, by giving them unity for a denominator. Thus, if 9 is to be multiplied by $\frac{4}{7}$, the operation becomes to multiply $\frac{9}{1}$ by $\frac{4}{7}$, which, according to the rule already given, produces $\frac{36}{7}$. This product is reduced to $5\frac{1}{7}$. *Hence, when a whole number is multiplied by a fraction, or a fraction by a whole number, the operation is reduced to multiplying the numerator of the fraction by the whole number.*

108. *If whole numbers are joined to fractions, before multiplication, the whole numbers must be reduced, each to the kind of fraction which accompanies it.* For example, if $12\frac{3}{5}$ is to be multiplied by $9\frac{3}{4}$, we change (86) the multiplicand into $\frac{63}{5}$, and the multiplier into $\frac{39}{4}$; and multiply $\frac{63}{5}$ by $\frac{39}{4}$ according to the rule in article (106), which gives $\frac{2457}{20}$, or $122\frac{17}{20}$.

EXAMPLES FOR PRACTICE.

1. Multiply $\frac{3}{8}$ by $\frac{4}{5}$. Ans. $\frac{3}{10}$.

Multiply $\frac{7}{9}$ by $\frac{2}{3}$. Ans. $\frac{14}{26}$.

3. Multiply $4\frac{1}{2}$ by $\frac{1}{8}$.	Ans. $\frac{9}{16}$.
4. Multiply $48\frac{3}{5}$ by $13\frac{5}{6}$.	Ans. $672\frac{3}{10}$.
5. Multiply $\frac{1}{2}$ of 7 by $\frac{3}{6}$.	Ans. $1\frac{3}{4}$.
6. Multiply $\frac{4}{15}$ by $\frac{5}{24}$.	Ans. $\frac{1}{18}$.
7. Multiply $\frac{25}{30}$ by $\frac{10}{12}$.	Ans. $\frac{25}{36}$.
8. Multiply $\frac{11}{13}$ by $\frac{31}{42}$.	Ans. $\frac{341}{546}$.

DIVISION OF FRACTIONS.

109. *In dividing one fraction by another, the terms of the fraction which is the divisor must be inverted; the fraction which is the dividend is then multiplied by the fraction thus inverted.*

For Example. To divide $\frac{4}{5}$ by $\frac{2}{3}$, the fraction $\frac{2}{3}$ is inverted and becomes $\frac{3}{2}$; we then multiply $\frac{4}{5}$ by $\frac{3}{2}$, according to the rule in article (106,) and have $\frac{12}{10}$ or $1\frac{2}{10}$ for the quotient of $\frac{4}{5}$ divided by $\frac{2}{3}$.

The reason of this rule may be seen by observing, that to divide $\frac{4}{5}$ by $\frac{2}{3}$ is to seek how many times $\frac{4}{5}$ contains $\frac{2}{3}$. But it is evident, since the divisor is 2 thirds, that it will be contained in the dividend three times as often as there are 2 whole numbers in it. It is then necessary first to divide by 2, and afterwards to multiply by 3; and this operation consists in taking three times the half of the dividend, or to multiply by $\frac{3}{2}$ which is the divisor inverted.

110. If a fraction is to be divided by a whole number, or a whole number by a fraction, the whole number must be written as a fraction by giving to it unity for a denominator. Thus if 12 is to be divided by $\frac{5}{7}$, the operation becomes to divide $\frac{12}{1}$ by $\frac{5}{7}$, and this, according to the rule, is to multiply $\frac{12}{1}$ by $\frac{7}{5}$, which gves $\frac{84}{5}$ or $16\frac{4}{5}$. In like manner, if the fraction $\frac{3}{4}$ is to be divided by 5, the operation becomes to divide $\frac{3}{4}$ by $\frac{5}{1}$, that is, to multiply $\frac{3}{4}$ by $\frac{1}{5}$, which gives $\frac{3}{20}$. Hence, *when a fraction is to be divided by a whole number, the operation requires the denominator to be multiplied by the whole number.*

111. If whole numbers are joined to fractions, the whole numbers must be reduced, each to the kind of fraction which accompanies it. For Example, if $54\frac{3}{4}$ is

to be divided by $12\frac{2}{3}$ we change the dividend into $\frac{273}{5}$, and the divisor into $\frac{38}{3}$. The operation then is to divide $\frac{273}{5}$ by $\frac{38}{3}$, that is, (109,) to multiply $\frac{273}{5}$ by $\frac{3}{38}$, which gives $\frac{819}{190}$, or $4\frac{59}{190}$.

EXAMPLES FOR PRACTICE.

1. Divide $\frac{4}{7}$ by $\frac{2}{3}$.	Ans. $\frac{6}{7}$.
2. Divide $3\frac{1}{2}$ by $9\frac{1}{2}$.	Ans. $\frac{1}{2}$.
3. Divide 5 by $\frac{7}{10}$.	Ans. $7\frac{1}{7}$.
4. Divide $\frac{9}{10}$ by $4\frac{1}{2}$.	Ans. $\frac{1}{5}$.
5. Divide $\frac{7}{8}$ by 4.	Ans. $\frac{7}{32}$.
6. Divide $3\frac{1}{6}$ by $9\frac{1}{2}$.	Ans. $\frac{1}{3}$.
7. Divide $9\frac{1}{6}$ by $\frac{1}{2}$ of 7.	Ans. $2\frac{13}{21}$.
8. Divide $\frac{2}{6}$ by $\frac{6}{12}$.	Ans. $\frac{2}{3}$.

APPLICATION OF THE PRECEDING RULES.

112. After what has been said in article (96) it is easy to see how the value of a fraction can be found. For example, if the value of $\frac{5}{7}$ of a pound is required; since $\frac{5}{7}$ of one pound is the same as the seventh of 5 pounds (96,) we reduce the 5 pounds to shillings (57,) and divide 100 shillings which is the result, by 7, which gives 14 shillings for the quotient and 2 shillings for remainder. The 2 shillings are reduced to pence, and we divide 24 pence by 7, and have 3 pence $\frac{3}{7}$. Thus the $\frac{5}{7}$ of a pound, are 14s. $3\frac{3}{7}$d.

If the $\frac{5}{7}$ of 24 pounds are required, we can, as in the last case, take the $\frac{5}{7}$ of one pound, and multiply 24 by that result; but it is more convenient to multiply, at first, $\frac{5}{7}$ by 24 pounds, which gives (107) $\frac{120}{7}$ of a pound. After estimating the value of this fraction, we find £17. 2s. $10\frac{2}{7}$d. for the answer.

The principle explained in this article may be expressed in the following rule for finding the.

VALUE OF A FRACTION IN THE KNOWN PARTS OF THE INTEGER.

Multiply the numerator by the number in the lower, which expresses an unit of the next higher denomination, and divide the product by the denominator. If there is a remainder, multiply it by the number in the lower, which expresses an unit in the next higher denomination; divide as before, and continue this operation as far as is necessary. The several quotients placed in their order will be the answer.

EXAMPLES FOR PRACTICE.

1. What is the value of $\frac{3}{8}$ of a pound sterling?
Ans. 7s. 6d.

2. What is the value of $\frac{3}{5}$ of a pound Troy?
Ans. 7oz. 4dwt.

3. What is the value of $\frac{4}{7}$ of a pound Avoirdupois?
Ans. 9oz. $2\frac{2}{7}$dr.

4. What is the value of $\frac{7}{9}$ of a cwt.?
Ans. 3qrs. 3lb. 10oz. $12\frac{4}{9}$drs.

5. What is the value of $\frac{3}{17}$ of a mile?
Ans. 1fur. 16pls. 2yds. 1ft. $9\frac{3}{17}$in.

6. What is the value of $\frac{5}{9}$ of an ell English?
Ans. 2qrs. $3\frac{1}{9}$nls.

7. What is the value of $\frac{7}{8}$ of a tun of wine?
Ans. 3hhds. 31gal. 2qts.

8. What is the value of $\frac{7}{13}$ of a day?
Ans. 12h. 55′ 23″$\frac{1}{13}$.

113. As decimal fractions have no denominator expressed, their value is more easily determined. If it is required, what is the value of 0,532 of a fathom; as the fathom is 6feet, we multiply 0,532 by 6, which gives 3,192feet; that is, 3ft. and 0,192 of a foot. Multiplying the last fraction by 12, to find the value in inches, we have 2,304 inches, or 2 inches and 0,304 of an inch. Multiplying this last fraction by 12, to reduce it to lines, we have 3,648 lines, or 3l. and 0,648 of a line. Hence the value 0,532 of a fathom is 3ft. 2in. 3l. and 0,648 of a line.

This principle is expressed in the following rule.

TO REDUCE A DECIMAL FROM A HIGHER TO A LOWER DENOMINATION,

" *Multiply it by so many as one makes of this lower, and those figures which remain on the left of the comma, when the proper number is separated for decimals, will constitute the whole number of this denomination, the decimal part of which may be still further reduced, if there be lower denominations, by multiplying it by the number which makes one of the next denomination; and so on.*"

EXAMPLES FOR PRACTICE.

1. What is the value of ,625 of a shilling? Ans. 7½s.
2. What is the value of ,37623 of a pound?
Ans. 7s. 6¼d.
3. What is the value of ,8322916 of a pound?
Ans. 16s. 7¾d.
4. What is the value of ,6725 of a cwt.?
Ans. 2qrs. 19lb. 5oz.
5. What is the value of ,67 of a league?
Ans. 2mls. 3pls. 1yd. 3in. 1bc.
6. What is the value of ,61 of a tun of wine?
Ans. 2hhd. 27gal. 2qt. 1pt.
7. What is the value of ,461 of a chaldron of coals?
Ans. 16bu. 2pks.
8. What is the value of ,42857 of a month?
Ans. 1w. 4d. 23h. 59′ 56″.

114. In determining the value of fractions we are naturally led to speak of *fractions* of *fractions*; which are a series of fractions separated by the particle *of*. For example, $\frac{2}{3}$ of $\frac{3}{4}$; $\frac{2}{3}$ of $\frac{3}{4}$ of $\frac{5}{6}$, &c. are fractions of fractions. *These compound fractions are reduced to single fractions, by multiplying all the numerators together for a new numerator, and all the denominators for a new denominator.* Thus, the fraction $\frac{2}{3}$ of $\frac{3}{4}$ is reduced to $\frac{6}{12}$, or $\frac{1}{2}$. The fraction $\frac{2}{3}$ of $\frac{3}{4}$ of $\frac{5}{6}$ is reduced to $\frac{30}{72}$ or $\frac{5}{12}$

If the value of $\frac{3}{4}$ of $5\frac{3}{8}$ is required, we change the whole number 5 into eighths, and the question becomes to find the value of the compound fraction $\frac{3}{4}$ of $\frac{43}{8}$, which is $\frac{129}{32}$, or $4\frac{1}{32}$

EXAMPLES FOR PRACTICE.

1. Reduce $\frac{2}{3}$ of $\frac{3}{4}$ of $\frac{4}{5}$ of $\frac{8}{11}$ to a single fraction. Ans. $\frac{16}{55}$.

2. Reduce $\frac{2}{3}$ of $\frac{3}{5}$ of $\frac{5}{8}$ to a single fraction. Ans. $\frac{1}{4}$.

3. Reduce $\frac{3}{4}$ of $\frac{2}{3}$ of $\frac{5}{6}$ of $\frac{4}{11}$ to a single fraction. Ans. $\frac{5}{33}$.

4. Reduce $\frac{11}{12}$ of $\frac{7}{13}$ of $\frac{8}{19}$ of 10 to a single fraction. Ans. $\frac{1540}{741}$.

This rule is also used for reducing fractions of one denomination to those of another, retaining the same value. For Example, if it is required to reduce $\frac{5}{6}$ of a penny to the fraction of a pound; we make a compound fraction of it, and write $\frac{5}{6}$ of $\frac{1}{12}$ of $\frac{1}{20}$, which becomes $\frac{5}{1440}$, or $\frac{1}{288}$.

5. Reduce $\frac{2}{3}$ of a farthing to the fraction of a pound. Ans. $\frac{1}{1440}$.

6. Reduce £$\frac{1}{18}$ to the fraction of a penny. Ans. $\frac{40}{3}$.

7. Reduce $\frac{2}{3}$ of a dwt. to the fraction of a pound Troy. Ans. $\frac{1}{360}$.

8. Reduce $\frac{6}{7}$ of a pound Avoirdupois to the fraction of a cwt. Ans. $\frac{6}{784}$

The following example embraces many of the rules, which have been established concerning fractions.

If it is required to construct a vessel whose length is $140\frac{2}{3}$ feet, and to have the sum of the spaces between the port-holes $108\frac{3}{4}$ feet including the distance between the first port-hole and the stem, and also the distance between the last port-hole and the stern post. To know if 12 port-holes can be pierced in each side of the vessel, we subtract $108\frac{3}{4}$ from $140\frac{2}{3}$ feet (103 and following), and there remains $31\frac{11}{12}$ feet for the port-holes. We then divide $31\frac{11}{12}$ by 12 (86) and (110), and have $\frac{383}{12}$ for the quotient. This quotient reduced to feet gives 2 feet and $\frac{95}{144}$ of a foot; which, reduced to inches and lines, gives 7 inches 11 lines. Each port-hole then may occupy the space of 2 feet, 7 inches, and 11 lines; which is the proper measure for a vessel whose length is $140\frac{2}{3}$ feet.

MISCELLANEOUS EXAMPLES.

1. Reduce $\frac{57}{13}$ to a mixed number.

Ans. $4\frac{5}{13}$.

2. Reduce $\frac{8753}{87}$ to a mixed number.

Ans. $10\frac{37}{45}$.

3. Reduce $\frac{3847}{784}$ to a mixed number.

Ans. $4\frac{711}{784}$.

4. Reduce $\frac{18006800}{24321}$ to a mixed number.

Ans. $740\frac{9260}{24321}$.

5. How many 7ths are there in 1? How many are there in 3? And how many in 13?

Ans. $1=\frac{7}{7}$. $3=\frac{21}{7}$, and $13=\frac{91}{7}$.

6. In $13\frac{7}{15}$ how many 15ths?

Ans. $13\frac{7}{15}=\frac{202}{15}$.

7. Reduce $7\frac{3}{57}$ to an improper fraction.

Ans. $\frac{402}{57}$.

8. Reduce $437\frac{2}{11}$ to an improper fraction.

Ans. $\frac{4809}{11}$

9. Reduce $63\frac{483}{2847}$ to an improper fraction.

Ans. $\frac{179784}{2847}$.

10. How much is 5 times $\frac{8}{17}$? Ans. $2\frac{6}{17}$.

11. How much is 7 times $\frac{58}{215}$? Ans. $1\frac{191}{215}$.

12. How much is 17 times $\frac{38}{275}$?

Ans. $2\frac{6}{275}$.

13. How much is 9 times $\frac{475}{2868}$?

Ans. $1\frac{1407}{2868}$.

14. How much is 35 times $\frac{215}{17834}$?

Ans. $\frac{7525}{17834}$.

15. Multiply $\frac{17}{2876}$ by 238. Ans. $1\frac{1170}{2876}$.

16. Multiply $\frac{28}{18407}$ by 1003.

Ans. $1\frac{9679}{18407}$.

17. Multiply $\frac{200}{17063}$ by 5060.

Ans. $59\frac{5283}{17063}$.

18. Multiply $\frac{16847}{1893700}$ by 607.

Ans. $5\frac{757629}{1893700}$.

19. If you divide 1 dollar equally among 23 persons, what do you give to each? If you divide $2, what does each receive? $7? $18? $34? $87? 253 dollars?

Ans. $\frac{1}{23}$ dol. $\frac{2}{23}$ dol., &c. $\frac{34}{23}=1\frac{11}{23}$, $\frac{87}{23}=3\frac{18}{23}$ dols. $\frac{253}{23}=11$ dols.

20. How many pence in $\frac{3}{4}$ of a shilling?

Ans. 9d.

21. How many farthings in $\frac{3}{5}$ of a penny?

Ans. $2\frac{2}{5}$qrs.

22. What is the value of $\frac{3}{5}$ of a shilling, in pence and farthings?

Ans. $7\frac{1}{5}$pence=7d. $\frac{4}{5}$qr.=$28\frac{4}{5}$qrs.

23. Find the value of $\frac{5}{7}$ of £1, in shillings, pence, and farthings? Ans. 14s. 3d. $1\frac{5}{7}$qrs.

24. What is the value of $\frac{4}{7}$ of a day, in hours, minutes, and seconds?

Ans. 13h. 42min. $51\frac{3}{7}$sec.

25. What is the value of $\frac{3}{8}$ of 1lb. Avoirdupois? Ans. 6oz.

26. What is $\frac{5}{17}$ of 1cwt.?

Ans. 1qr. 4lb. $15\frac{1}{17}$oz.

27. What is $\frac{5}{18}$ of 1hhd. of wine?

Ans. 17galls. 2qts.

28. What is $\frac{4}{7}$ of a yard? Ans. 2qrs. $1\frac{1}{7}$nls.

29. What is $\frac{5}{28}$ of a dollar?

Ans. $0, $17,8\frac{16}{28}$.

30. What is $\frac{14}{37}$ of £1? Ans. 7s. 6d. $3\frac{9}{37}$qrs.

31. What is $\frac{38}{57}$ of a day? Ans. 16hours.

32. What is $\frac{43}{53}$ of a yard? Ans. $3\frac{13}{53}$qrs.

33. What is $\frac{18}{43}$ of a bushel?

Ans. 1pk. $5\frac{17}{43}$qts.

34. What is $\frac{248}{684}$ of 1hhd. of brandy?

Ans. 22gals. $3\frac{252}{684}$qts.

35. If $\frac{3}{5}$ of a barrel of flour were to be divided equally among 3 men, what would each receive? Ans. $\frac{1}{5}$ of a barrel.

36. What is $\frac{1}{2}$ of $\frac{1}{3}$? Ans. $\frac{1}{6}$.

37. What is $\frac{1}{3}$ of $\frac{2}{9}$? Ans. $\frac{2}{27}$.

38. During a storm, a master of a vessel was obliged to throw overboard $\frac{4}{15}$ of the whole cargo. What loss must a man sustain who owned $\frac{3}{8}$ of the cargo? Ans. $\frac{12}{120}$ or $\frac{1}{10}$.

39. A man owned $\frac{3}{25}$ of the capital in a cotton manufactory, and sold $\frac{4}{11}$ of his share. What part of the whole capital did he sell? And what part did he still own? Ans. he sold $\frac{12}{275}$. At first, he owned $\frac{3}{25}$ of the whole. $\frac{1}{25}=\frac{11}{275}$ and $\frac{3}{25}=\frac{33}{275}$; of these he sold $\frac{12}{275}$, consequently, had $\frac{21}{275}$ left. He therefore sold $\frac{12}{275}$, and had $\frac{21}{275}$ left.

40. Divide $\frac{24}{125}$ by 6. Ans. $\frac{4}{125}$.

Divide $\frac{53}{87}$ by $\frac{1}{25}$. Ans. $\frac{53}{2175}$.

41. What part of 8 is $7\frac{2}{3}$? Ans. $\frac{23}{24}$.

42. What part of 19 is $14\frac{7}{8}$? Ans. $\frac{119}{152}$.

43. Divide $15\frac{23}{33}$ by $\frac{1}{8}$. Ans. $\frac{47}{1064}$.

44. A lady being asked her age, said that her husband was $37\frac{5}{9}$ years old, and she was not so old as her husband by $8\frac{9}{13}$ years. What was her age? Ans. $28\frac{101}{117}$ years.

45. Add together $\frac{4}{5}$, $\frac{2}{7}$, and $\frac{3}{4}$. Ans. $1\frac{117}{140}$.

46. Add together $137\frac{2}{5}$, $26\frac{9}{15}$, and $243\frac{2}{7}$.

Ans. $407\frac{2}{7}$.

47. Subtract $38\frac{4}{19}$ from $53\frac{3}{25}$. Ans. $14\frac{432}{475}$.

48. Subtract $284\frac{38}{175}$ from $813\frac{21}{97}$.

Ans. $528\frac{16728}{16781}$.

49. Reduce $\frac{486}{9720}$ to its lowest terms. Ans. $\frac{1}{20}$.

50. Reduce $\frac{4746}{38233}$ to its lowest terms.

Ans. $\frac{1582}{12811}$.

51. Reduce $\frac{5}{18}$, $\frac{8}{27}$, and $\frac{17}{90}$, to their least common denominator. Ans. $\frac{75}{270}$, $\frac{80}{270}$, $\frac{51}{270}$.

52. Two men hired a pasture for 21 dollars. One kept his horse in it $5\frac{1}{2}$ weeks, and the other $7\frac{2}{3}$ weeks; what ought each to pay? $5\frac{1}{2}=\frac{11}{2}=\frac{33}{6}$; and $7\frac{2}{3}=\frac{23}{3}=\frac{46}{6}$. $33+46=79$. The first ought to pay $\frac{33}{79}$, and the second $\frac{46}{79}$ of 21 dollars. Ans. \$$8\frac{61}{79}=8,77\frac{17}{79}$, and \$$12\frac{18}{79}=12,22\frac{62}{79}$.

53. A man failing in trade is able to pay only $\frac{13}{15}$ of a pound on a pound. How much can he pay on £17 15s. Ans. $15\frac{115}{300}$=£15 7s. 8d.; for £17 15s.=£$17\frac{15}{20}$= £$\frac{355}{20}$; now if $\frac{13}{15}$ be multiplied by $\frac{355}{20}$, we have £$15\frac{115}{300}$.

54. At \$4 a yard, what part of a yard of cloth can be bought for $\frac{1}{3}$ of a dollar?

Ans. $\frac{1}{12}$ of a yard.

55. If 3 sheep are worth $23\frac{2}{5}$ bushels of wheat, how many bushels is 1 sheep worth? And how many bushels are 50 sheep worth at that rate? Ans. $7\frac{4}{5}$bu. 390bu.

56. If 9 acres 1 rood produce 136 bushels of [illegible], what part of a rood will produce 1 bushel? [illegible]d how many acres will produce 500 bushels? Ans. $\frac{37}{136}$ of a rood; and $\frac{37}{136}\times500=\frac{17500}{136}=136\frac{4}{136}$ roods=34 acres, $0\frac{4}{136}$ roods.

57. If it take $1\frac{19}{24}$ yard of cloth to make a coat, how much will it take to make 8 coats, and how much to make 24 coats?

Ans. $14\frac{1}{3}$yds. 43yds.

58. There is a pole standing, so that $\frac{4}{7}$ of it is in the water, $\frac{3}{5}$ as much in the mud as in the

water, and $7\frac{3}{8}$ feet of it above the water. What is the whole length of the pole? The fractions $\frac{3}{7}$ and $\frac{2}{5}$ are more convenient than $\frac{4}{7}$ and $\frac{3}{5}$. There is $\frac{4}{35}$ of it in the mud; and in the mud and water both, $\frac{14}{35}$ of it; therefore $7\frac{3}{8}$ is $\frac{21}{35}$ of the whole pole. Ans. $12\frac{7}{24}$ft.=12ft. $3\frac{1}{2}$in.

In articles (90) and (91) rules were given for reducing fractions to a common denominator; but the denominator thus found is not always the smallest possible common denominator. For example, if it is proposed to reduce the fractions $\frac{7}{12}$ and $\frac{5}{8}$ to the same denominator, we need only multiply the terms of the first by 2, and of the second by 3, and they will have 24 for a common denominator, which is less than 96 given by the general rule. In this example, the reason why 24 will serve for the common denominator, is easily perceived; for if the operation be performed in the usual manner, and the fractions reduced to their simplest expressions, 24 will be the smallest possible common denominator. But when there is a large number of fractions in the question, it is impossible to discover immediately the smallest possible common denominator. *We will therefore give a general method for finding the smallest common denominator, when many fractions are in the given question.* It may, however, be first observed, that no changes can be made in a fraction without changing its value, except by division or multiplication. We shall employ no operations but these. The first is useful in reducing the given fractions to their simplest expression; the second is then used for finding the smallest common denominator. To explain the method we design to pursue, let the fractions $\frac{14}{60}$, $\frac{40}{336}$, and $\frac{65}{630}$, be considered. In order to reduce these fractions to their most simple expression, we decompose the numerators and denominators into their prime factors; and then consider what factors are common to both terms, and suppress them. The fractions thus decomposed are:

$$\frac{\dot{2} \text{ by } 7}{\dot{2} \text{ by } 2 \text{ by } 3 \text{ by } 5}$$

$$\frac{\dot{2} \text{ by } \dot{2} \text{ by } \dot{2} \text{ by } 5}{\dot{2} \text{ by } \dot{2} \text{ by } \dot{2} \text{ by } 2 \text{ by } 3 \text{ by } 7}$$

$$\frac{\dot{3} \text{ by } \dot{5}}{2 \text{ by } \dot{3} \text{ by } \dot{5} \text{ by } 11}$$

On suppressing all the prime factors, which are common to both terms, the fractions are reduced to

$$\frac{7}{2 \text{ by } 3 \text{ by } 5}, \quad \frac{5}{2 \text{ by } 3 \text{ by } 7}, \quad \frac{1}{2 \text{ by } 11}$$

As the denominators of these fractions are the simplest possible, it is plain, they cannot have a smaller common denominator than 2310, a number formed by the product of the prime factors 2, 3, 5, 7, 11, which are the differences between the denominators.

Thus, the denominator 11 being contained in one of these fractions, we cannot give to this fraction a denominator common to the others, unless both its terms are multiplied by the same number. The common denominator will then necessarily contain the factor 11; and as the same may be said of the factors 2, 3, 5, 7, it follows, that the common denominator necessarily contains each of the factors 2, 3, 5, 7, 11. It cannot then be smaller than their product 2310. Having thus obtained, 2310, the smallest common denominator, we next reduce the fractions $\frac{7}{2 \text{ by } 3 \text{ by } 5}$, $\frac{5}{2 \text{ by } 3 \text{ by } 7}$, $\frac{1}{2 \text{ by } 11}$, to this common denominator, by multiplying both terms of each of these fractions by the product of the prime factors, which were wanting in the denominator of each to complete the smallest common denominator 2310. They then become

$$\frac{7}{2 \text{ by } 3 \text{ by } 5} \cdot \frac{\textit{by } 7 \textit{ by } 11}{\textit{by } 7 \textit{ by } 11} \text{ or } \frac{539}{2310}.$$

$$\frac{5}{2 \text{ by } 3 \text{ by } 7} \cdot \frac{\textit{by } 5 \textit{ by } 11}{\textit{by } 5 \textit{ by } 11} \text{ or } \frac{275}{2310}.$$

$$\frac{1}{2 \text{ by } 11} \cdot \frac{\textit{by } 3 \textit{ by } 5 \textit{ by } 7}{\text{by } 3 \text{ by } 5 \text{ by } 7} \text{ or } \frac{105}{2310}.$$

The proposed fractions $\frac{14}{60}$, $\frac{40}{336}$, $\frac{65}{330}$, have now become by reducing them to their smallest common divisor, $\frac{539}{2310}$, $\frac{275}{2310}$, $\frac{105}{2310}$. The common denominator 2310 is also much smaller than 6652800, which would have been obtained by following the rule in article (91.)

If we generalize the reasoning which has conducted to this result, the following general rule may be deduced.

TO REDUCE MANY FRACTIONS TO THEIR SMALLEST POSSIBLE COMMON DENOMINATOR.

1. *Reduce the proposed fractions to their most simple expressions, by decomposing their terms into their prime factors, and suppressing those factors which are common to both terms.*

2. *Take, among the prime factors, which compose the denominators of the fractions thus reduced, those which are different; the product of these last factors will be the smallest possible common denominator.*

It may be well to observe that, *if one of the prime factors is many times factor in any denominator; it will be as many times factor in the common denominator, as it is in the denominator, in which it is found the greatest number of times.* The reason for this is the same as has already been given.

115. When in the course of a calculation, we are conducted to a fraction whose numerator and denominator are considerably large, and have no common factor; in order to form a more exact idea of it, we seek the approximate values of this fraction, expressed in more simple numbers. If, for instance, we have the fractional number $\frac{1103}{887}$, the whole number, which is 1, is first sought, and the resulting fraction is $\frac{216}{887}$. Now, to form a more simple idea of this fraction, we seek to compare a part of it with unity, that only one term may be considered. For this purpose, both terms of the fraction are divided by 216, and 1 is found for the quotient of the numerator, and $4\frac{23}{216}$ for that of the denominator. This last quotient, contained between 4 and 5, shows that the value of the fraction $\frac{216}{887}$ is between $\frac{1}{4}$ and $\frac{1}{5}$. By staying the operation at this point, we perceive, that the second ap-

proximate value of the fraction $\frac{1103}{887}$ is 1 and $\frac{1}{4}$, or $\frac{1}{5}$. But this value is too great, because the true value will be equal to 1 plus 1 divided by 4 and $\frac{23}{216}$. It is therefore written thus: $1\frac{1}{4\frac{23}{216}}$. To form an exact idea of the expression $\frac{1}{4\frac{23}{216}}$, we must consider it as indicating the quotient of the whole number 1 divided by the whole number 4 accompanied by the fraction $\frac{23}{216}$. If both terms of $\frac{23}{216}$ are divided by 23, the quotient will be $\frac{1}{9\frac{9}{23}}$, neglecting the $\frac{9}{23}$, which accompany the whole number 9, we shall have only $\frac{1}{9}$ instead of $\frac{23}{216}$; consequently, $1\frac{1}{4\frac{1}{9}}$ will be the third approximate value of $\frac{1103}{887}$, which is too small; because 9 being less than the true quotient of 216 divided by 23, the fraction $\frac{1}{9}$ will be greater than that, which ought to accompany 4. Consequently, the divisor $4\frac{1}{9}$ will be greater than the exact divisor $4\frac{23}{216}$; and the quotient $\frac{1}{4\frac{1}{9}}$ will be smaller than the true quotient. By reducing the whole number 4 and the fraction accompanying it, and performing the division according to the direction given in articles (86) and (109), we obtain $\frac{9}{37}$; and have 1 and $\frac{9}{37}$, or $\frac{46}{37}$, for the third approximate value of $\frac{1103}{887}$. The exact expression of this value being $1\cfrac{1}{4\cfrac{1}{9\frac{9}{23}}}$, if we divide the two terms of $\frac{9}{23}$ by 9, we shall have $1\cfrac{1}{4\cfrac{1}{9\cfrac{1}{2\frac{5}{9}}}}$; neglecting the fraction $\frac{5}{9}$, there will remain $1\cfrac{1}{4\cfrac{1}{9\cfrac{1}{2}}}$, because the fraction $\frac{1}{2}$ being greater than $\frac{1}{2\frac{5}{9}}$ whose place it occupies, will form, when joined to 9, a denominator too large. The fraction joined to 4 will then be too small; and the last denominator being too small, will make the last fraction too great.

By reducing $9\frac{1}{2}$ to a fraction, we have $\frac{19}{2}$; $\frac{1}{9\frac{1}{2}}$ will then be $\frac{2}{19}$, and the approximate value will become $1\cfrac{1}{4\frac{2}{19}}$; but $\frac{1}{4\frac{2}{19}}$ gives $\frac{19}{78}$. By adding unity to this fraction it becomes $1\frac{19}{78}$, or $\frac{97}{78}$ for the fourth approximate value of $\frac{1103}{887}$. Resuming the expression, $1\cfrac{1}{4\cfrac{1}{9\cfrac{1}{2\frac{5}{9}}}}$, we divide the two terms of the last fraction $\frac{5}{9}$ by 5, and obtain $\frac{1}{1\frac{4}{5}}$ and $1\cfrac{1}{4\cfrac{1}{9\cfrac{1}{2\cfrac{1}{1\frac{4}{5}}}}}$; neglecting the fraction $\frac{4}{5}$, there remains $1\cfrac{1}{4\cfrac{1}{9\cfrac{1}{2\frac{1}{1}}}}$; it may be seen, as above, that this value is smaller than the true value.

The fraction $\frac{1}{2\frac{1}{1}}$ becomes reduced to $\frac{1}{3}$; and as the preceding $\frac{1}{9\frac{1}{3}}$ gives $\frac{3}{28}$, the next preceding becomes $\frac{1}{4\frac{3}{28}}$, equal to $\frac{28}{115}$. The fifth approximate value is then $1\frac{28}{115}$ or $\frac{143}{115}$.

Lastly, the two terms of the fraction $\frac{4}{5}$, which was neglected above, are divided by 4, and $\frac{1}{1\frac{1}{4}}$ is the quotient. By suppressing the fraction $\frac{1}{4}$, we obtain the new value $1\cfrac{1}{4\cfrac{1}{9\cfrac{1}{2\cfrac{1}{1\frac{1}{1}}}}}$, greater than the true value. By reducing all these denominators successively to a fraction, we shall find, that the simple fraction, which represents them, is $\frac{240}{193}$ or $1\frac{47}{193}$. When the fraction $\frac{1}{4}$ is restored to the side of the last denominator, we form the expression $1\cfrac{1}{4\cfrac{1}{9\cfrac{1}{2\cfrac{1}{1\cfrac{1}{1\frac{1}{4}}}}}}$, which being reduced as the preceding reproduces the fractional number $\frac{1103}{887}$. By operating in the same manner upon any other fraction, we may obtain a series of approximate values, alternately less and greater than the true value; if as in the preceding case the numerator exceeds the denominator: or greater and less, if the denominator is greater than the numerator.

The developements for the fraction $\frac{1103}{887}$ are called *continued fractions*, which may be defined; *Fractions whose denominator is composed of a whole number and a fraction, which fraction has also a whole number and a fraction for its denominator; and so on.*

Note. The method explained in the preceding article serves for the summation of the continued fraction in article (261) of Legendre's Geometry, used in the Cambridge Course of Mathematics.

COMPLEX NUMBERS.

116. The rules already explained might serve for calculating complex numbers; it may, however, be proper to consider them in a more particular manner, as the division, which they make of the principal unit, often facilitates their calculation.

ADDITION OF COMPLEX NUMBERS.

117. *In this operation, the proposed numbers are written under each other in such a manner, that all the parts of the same name are in the same vertical column. A line is then drawn under the whole; and the addition commenced with the parts of the lowest kind. If their sum does not compose an unit of the next higher denomination, this sum is written under the units of its own kind; but if it contain parts enough to compose one, or more units of the next higher denomination, we write what remains over an exact number of units of this second sort, and retain these units to be added with those of their own kind in the next column. Proceed in the same manner through the several denominations in the sum.*

EXAMPLE.

It is proposed to add	£227	14s.	8d.
	2549	18	5
	184	11	11
	17	10	7
	£2976	15s.	7d. sum.

The sum of the pence is 31, but 12 pence make a shilling; therefore 31 pence contain 2 shillings and 7 pence. We write 7 under the column of pence and retain 2 to be added with the shillings. The units in the first column of shillings are 13, to which add the 2 which were retained from the pence and we have 15, of which we write 5 only under the shillings, and retain the ten to be added to the tens in the second column of shillings. The amount of this column with the 1 added to it is 5; and as 2 tens of shillings make a pound, we take the half of 5 which is 2, with 1 for remainder. The remainder is written under the column of tens in the shillings, and the 2 carried to be added with the pounds in the usual manner.

EXAMPLES.

£	s.	d.
17	13	4
13	10	2
10	17	3
8	8	7
3	8	4
	8	8
54	6	4 sum.

£	s.	d.
175	12	6
280	10	4
362	8	3
484	13	10
40	18	11
1344	3	10

£	s.	d.
456	12	5½
320	12	0
400	10	1¾
45	6	2¼
1000	18	3¼
2223	19	0¾

TROY WEIGHT.

℔	oz.
37	11
62	0
72	10
13	4

℔	oz.	dwt.
93	11	18
6	0	1
14	4	12
72	11	3

℔	oz.	dwt.	gr.
17	9	11	15
82	2	0	0
12	8	16	0
9	4	0	0

AVOIRDUPOIS WEIGHT.

T.	cwt.	qrs.
40	11	3
15	10	2
16	0	1
9	12	0

T.	cwt.	qrs.	℔.
3	19	3	27
2	12	1	12
10	5	0	2
	3	2	10

cwt.	qr.	℔.	oz.	dr.
8	3	12	15	8
12	0	2	9	4
8	2	1	0	3
9	1	26	8	2

LONG MEASURE.

L.	Mi.	Fur.
7	2	7
6	1	4
12	1	0
14	0	3

L.	Mi.	Fur.	pls.
8	2	7	16
20	2	7	10
24	1	4	9
50	0	0	0

yd.	ft.	in.	lin.
2	2	9	9
1	1	0	0
10	1	3	7
12	0	4	0

LAND MEASURE.

A.	R.	pls.
75	3	2
24	0	0
98	1	0
75	3	6

A.	R.	pls.
150	3	39
265	2	11
204	1	0
326	0	0

yd.	ft.
6	6
10	4
12	3
20	0

EXAMPLES FOR PRACTICE.

1. Bought an English Reader for 5s. 7½d.; a Testament for 6s. 6¼d.; an English Grammar for 3s. 9d.; and a Slate for 2s. 4d. What did the whole come to?
Ans. 18s. 3d.

2. If a merchant buy cloth to the amount of £310 7s. 6d.; linen to the amount of £37 5s.; and groceries to the amount of £209 15s. 4½d. What sum must he pay for the whole? Ans. £557 7s. 10½d.

3. Bought of a silversmith dishes weighing 16℔. 10oz. 13dwt.; plates weighing 35℔. 10oz. 11dwt.; table spoons, 6℔. 11oz.; and tea spoons, 2℔. 8oz. What was the weight of the whole? Ans. 62℔. 4oz. 4dwt.

4. A grocer bought 4 hogsheads of sugar, which weighed as follows,—No. 1, 8cwt. 1qr. 26℔.; No. 2, 9cwt. 3qrs. 11℔.; No. 3, 12cwt. 2qrs. 19℔.; No. 4, 12cwt. 2 qrs. What did the whole weigh?

Ans. 43cwt. 2qrs.

5. A man travelled in one day 27miles, 2furlongs; in another, 32mi. 7fur. 33rods; in another, 19mi. 7fur. 16rods; in another, 12mi. 5fur. How far did he travel in all? Ans. 92mi. 6fur. 9rods.

6. There are four pieces of linen; the first contains 27yds. 2qrs.; the second, 41yds. 3qrs.; the third, 36yds. 1qr.; the fourth, 33yds. 2qrs. How many are there in the four pieces? Ans. 139 yards.

7. Sold two casks of cider, one of which contained 31 gallons, 3 quarts; and the other, 36 gallons, 2 quarts, 1 pint. How much was there in the two?

Ans. 68gall. 1qt. 1pt.

The method of proving the addition of complex numbers, is founded on the same principles as that for simple numbers, and is performed in the same manner; but we must observe in passing from a higher to a lower denomination, that the rate of decrease is not by ten, but by such a number of the lower as expresses an unit in the next higher denomination.

SUBTRACTION OF COMPLEX NUMBERS.

118. *Write the proposed numbers as in addition, and begin subtraction with the units of the lowest kind. If the lower can be subtracted from the upper number, write the remainder underneath. But if it cannot be subtracted, borrow an unit from the next higher denomination; reduce, and add it to the kind of units from which you subtract. Perform the same operation upon the several denominations, and when you have borrowed, diminish the number from which you borrow by an unit; and write each remainder under the number from which it is taken.*

EXAMPLE.

From	£143	17s.	6d.	
Take	£75	12s.	9d.	
	£68	4s,	9d.	rem.

Not being able to take 9d. from 6d., we borrow 1s. which is 12d. and add 12 to 6, which make 18; from which taking 9, there remains 9. We then take 12s. not from 17s. but from 16 which remain after 1 was borrowed, and 4 remains. Finally, we subtract £75 from £143 and £68 remain.

EXAMPLE II.

From	£163	0s.	5d.
Take	£84	18s.	9d.
	£78	1s.	8d. rem.

As we cannot take 9d. from 5d., and as there are no shillings from which we can borrow, we borrow 1 pound from 163 pounds, but suppose that 19, of the 20 shillings thus borrowed, are left in the place of the cipher; we then operate as before.

TROY WEIGHT.

	℔.	oz.	℔.	oz.	dwt.	℔.	oz.	dwt.	gr.
From	48	2	15	5	2	45	9		3
Take	10	1	12	0	2	15	6	18	17
Rem.									

AVOIRDUPOIS WEIGHT.

	T.	cwt.	qr.	T.	cwt.	qr.	℔.	cwt.	qr.	℔.	oz.	dr.
From	45	11	3	52	12	3	15	17	0	0	0	0
Take	15	10	2	24	10	0	26	10	3	4	3	2
Rem.												

LONG MEASURE.

	L.	Mi.	Fur.	L.	Mi.	Fur.	li.	yds.	ft.	in.	rds.
From	24	1	7	56	1	0	19	6	2	10	9
Take	18	2	4	10	4	7	20	3	2	7	4

CLOTH MEASURE.

	yds.	qrs.	na.	E.E.	qrs.	na.	E.Fl.	qrs.	na.
From	71	3	1	42	0	2	51	2	2
Take	14	2	3	19	2	3	42	2	1
Rem.									

LAND MEASURE.

	A.	R.	pls.	A.	R.	pls.	yds.	ft.
From	96	3	36	195	2	2	25	2
Take	25	2	39	36	3	1	14	7
Rem.								

TIME.

	Y.	mo.	W.	D.	H.	D.	H.	min.	sec.
From	75	3	32	6	20	36	14	30	25
Take	25	4	12	4	22	15	12	25	32
Rem.									

EXAMPLES FOR PRACTICE.

1. If a merchant buy a quantity of tobacco for £1500 16s., and afterwards sell it for £1595 19s. 9d.; how much will he gain by the transaction?

Ans. 95 3s. 9d.

2. A silversmith had 26℔. 9oz. 10dwt. of silver, and sold 18℔. 16dwt. 10grs. How much had he left?

Ans. 8℔. 8oz. 13dwt. 14grs.

3. A grocer has 13cwt. 2qr. 16℔. of sugar. If he sell 9cwt. 2qr. 7℔., how much will remain unsold?

Ans. 4cwt. 9℔.

4. There is a quantity of spice, which, with the box that contains it, weighs 34℔. 10oz. 1dr.; the box itself weighs 10℔. 10oz. 2dr. What is the weight of the spice?

Ans. 23℔. 15oz. 15dr.

5. A certain rope is 365yds. 1ft. 6in. long. If 84yds. 2ft. 4in. be cut off from it, how long will be the remainder?

Ans. 280yds. 2ft. 2in.

6. Bought 145yds. 3qrs. of cloth, and sold 95yds. 2qrs. 3na. of it; how much was left?
Ans. 50yds. 1na.

7. A farmer had 450A. 3R. of land; and gave his son 150A. 3R. 25pls. How much had he remaining?
Ans. 299A. 39R. 15pls.

8. Out of a granary which contained 500 bushels of wheat, there was taken 374bush. 2pks. 7qts. What quantity remained? Ans. 125bu. 1pk. 1qt.

9. Charles was bound an apprentice for 7 years, and has served 2 years and 5 months. How long has he still to serve? Ans. 4years, 7mo.

The subtraction of complex numbers, like simple subtraction, is proved by adding the difference to the less number, which, if the operation has been correctly performed, will reproduce the greater number.

MULTIPLICATION OF COMPLEX NUMBERS.

119. *The multiplication of complex numbers may be reduced, in general, to the multiplication of a fraction by a fraction.* The rule for this multiplication has been given in article (106.) For example, if it is required what 54 fathoms and 3 feet of work come to at £42, 17s. 8d. for a fathom; we can reduce £42, 17s. 8d. the multiplicand to pence (57), which gives 10292 pence. And as the penny is the 240th part of a pound, the multiplicand may be represented by $\frac{10292}{240}$ of a pound. In like manner, the multiplier 54F. 3ft. may be reduced to feet, which gives 327ft. And as the foot is the sixth part of the fathom, we can have $\frac{327}{6}$ of a fathom, for the multiplier. The question then becomes to multiply $\frac{10292}{240}$ of a pound by $\frac{327}{6}$, which gives (106) $\frac{3365484}{1440}$ of the pound, which (112) has the value £237, 2s. 10d.

This method of calculation may be applied to all kinds of complex numbers.

120. A number, which is contained in another an exact number of times, is called an *aliquot* part of that other number. Thus, 3 is an aliquot part of 12; 2, 4, and 6 are also aliquot parts of the same number.

Let us recollect, that multiplication consists in taking the multiplicand a certain number of times. For example, to multiply by 8¾, is to take the multiplicand 8 times and then to take the ¾ of it. But we can take these ¾, either by taking at first the fourth part of the multiplicand and writing it three times, or by taking at first the half, and afterwards the half of this half. Thus, to multiply 84 by 8¾, we write 84

8¾

672

42

21

---

735 product.

After multiplying 84 by 8, we have 672 for the product. Then to take ¾ of 84, we first take the half, which is 42, and for the remaining fourth, we take the half of 42, which is 21; and adding these several products, have 375 for the total product.

121. To apply this reasoning to complex numbers, we may remark, *that the different kinds of units below the principal unit, are fractions of each other and of that unit.* Consequently, to multiply with facility by these kinds of numbers, it is necessary to decompose them into aliquot parts of the principal unit, in such manner, that the aliquot parts may be conveniently employed,—or to decompose them into aliquot parts of each other. And if this decomposition does not furnish such aliquot parts as are convenient to use in calculation, they must be supplied by false products. The following examples will explain this method of operation.

EXAMPLE I.

It is required to know how much 54 fathoms 3ft. of work would cost at £72 for the fathom?

It is necessary to multiply £72

by 54F. 3ft.

£288 0s. 0d.

360

36

£3924 0s. 0d.

We multiply, at first, according to the common rules, £72 by 54. Then to multiply by 3ft. which are the half of a fathom, and consequently ought to cost half the price of it, we take the half of £72, and adding it to the total product, have £3924.

EXAMPLE II.

If we have £72
To be multiplied by 54F. 5ft.

£288 0s. 0d.
360
36
24

£3948 0s. 0d.

We first multiply 72 pounds by 54. Then instead of multiplying by $\frac{5}{6}$, because 5 feet make $\frac{5}{6}$ of a fathom, we decompose 5 feet into 3ft. and 2ft., of which, the first is half, and the second $\frac{1}{3}$ of a fathom. We first take $\frac{1}{2}$ of £72, and then $\frac{1}{3}$ of the same quantity, and have, by adding the several products, 3948 pounds for the total product.

EXAMPLE III.

When we have £72
To be multiplied by 5F. 4ft. 8in.

£360 0s. 0d.
36
12
4
4

£416 0s. 0d.

After having multiplied by 5F. we multiply by 4ft.; and for this purpose, decompose the number 4 into 3ft. and 1ft.; for 3ft. we take half of £72, which is £36, and for 1ft., we observe, that it is $\frac{1}{3}$ of 3ft., consequently, we take $\frac{1}{3}$ of £36, which is £12. Then, to multiply by 8 inches, instead of comparing these 8 inches with the fathom, we compare them with the foot, and decompose them into 4 inches and 4 inches, which are each $\frac{1}{3}$ of a foot, and which, consequently, will each give the $\frac{1}{3}$ of 12 pounds. After uniting these separate products, we have £416, 0s. 0d. for the total product.

122. If the multiplicand is also a complex number, the operation is conducted as will be explained in the following example.

EXAMPLE IV.

If we have	£72	6s.	6d.
To multiply by	27F.	4ft.	8in.
	£504	0s.	0d.
	144		
	6	15	0
	1	7	0
	0	13	6
	36	3	3
	12	1	1
	4	0	4$\frac{1}{3}$
	4	0	4$\frac{1}{3}$
	£2009	0s.	6$\frac{2}{3}$d.

We first multiply 72 pounds by 27. Then to multiply 6s. by 27, we decompose the 6s. into 5s. and 1s. The 5s. making a fourth part of the pound, ought, being multiplied by 27, to give 27 times the fourth of a pound, or the fourth of 27 pounds. We then take the fourth of £27, which is £6 15. To multiply by 1s. we observe, that 1s. is the fifth part of 5s. which have been already multiplied; we therefore take the fifth part of £6 15s., which is £1 7s.

Of the 6 pence we observe, that they are the half of a shilling; consequently, we take the half of £1 7s. the product for 1 shilling. Thus the whole multiplicand has been multiplied by 27.

To multiply by 4 feet, we take, (as in the preceding example,) for the 4 feet, first the half, £36 3s. 3d. of the multiplicand; and for 1 foot, the third of what 3 feet gave.

Lastly, to multiply by 8 inches, we take for 4, twice, that is, we twice write the third of what was given as a product for 1 foot; and adding these several parts, we have 2009 pounds, 0s. 6$\frac{2}{3}$d. for the total product.

123. In the preceding examples, the aliquot parts of

the multiplicand have been easily found; but in cases where the parts are more complicated, the process is conducted as in the following example.

EXAMPLE V.

At the price £34 10s. 2d. a Toise or fathom,
what ought 17 T. or F. to cost?

£238 0s. 0d.
34
8 10
. ..
0 17
0 2 10

£586 12s. 10d.

After having multiplied £34 by 17, and then 10s. by 17, taking half of 17 for the product, we multiply the 2 pence which are the sixth part of a shilling, and consequently, the sixth part of the tenth part, or (114) $\frac{1}{60}$ part of 10 shillings; but instead of taking the $\frac{1}{60}$ part of £8 10s. it will be more convenient to make a false product, and take at first the tenth of what 10s. have given, that is the tenth part of £8 10s.; this tenth, which is £0 17s., is a product for 1 shilling. But as we require the product for only $\frac{1}{6}$ of a shilling, the false product is crossed out, and we write the sixth underneath.

EXAMPLE VI.

For £34 10s. 2d. how much work should be done at the price of £1 for 17 Fathoms?

It is necessary to multiply 17 Fathoms by £34 10s. 2d. that is, to take 17 Fathoms as many times as 1 pound is contained in £34 10s. 2d.

17F.
£ 34 10s. 2d.

68F. 0ft. 0in. 0l. 0pts.
51
8 3
. . . . ..
0 5 1 2 $4\frac{4}{5}$
0 0 10 2 $4\frac{4}{5}$

586F. 3ft. 10in. 2l. $4\frac{4}{5}$pts.

We first multiply 17F. by 34; then to multiply 17 fathoms by 10 shillings, we take the half of 17 fathoms, because 10 shillings are the half of a pound, and have 8 fathoms and 3 feet. To multiply by 2 pence, we seek, for greater facility in the operation, what the product is for 1 shilling, by taking the tenth of what 10 shillings have given. This tenth is 0 fathoms, 5 feet, 1 inch, 2 lines, 4 points and $\frac{8}{10}$ or $\frac{4}{5}$ of a point; but as it should make no part of the product, we cross it out; taking, however, the sixth part of it for the product of 2 pence. This sixth part, which is 0 fathoms, 0 feet, 10 inches, 2 lines, 4 points and $\frac{24}{30}$ or $\frac{4}{5}$, is written underneath. On adding these separate parts, the total product is 586F. 5ft. 10in. 2l. 4$\frac{4}{5}$pts.

EXAMPLES FOR PRACTICE.

1. What will 7 barrels of flour come to at £2 16s. 6d. per barrel? Ans. £19 15s. 6d.

2. What will 6 pounds of tea come to at 5s. 3d. per pound? Ans. £1 11s. 6d.

3. What is the weight of 6 barrels of sugar, each weighing 1cwt. 3qrs. 20lb.? Ans. 11cwt. 2qrs. 8lb.

4. What is the weight of 6 chests of tea, each weighing 3cwt. 2qrs. 9lb.? Ans. 21cwt. 1qr. 26lb.

5. If 8El. E. 3qrs. 3na. of broadcloth will make a suit of clothes; how much of the same cloth will make 12 similar suits? Ans. 105 El. Eng.

6. How far will a man travel in 7 days, if he go 31mi. 31pls. 6ft. 6in. each day?
Ans. 217mi. 5fur. 19pls. 2ft. 6in.

7. In 9 fields containing 14 acres, 1 rood, and 25 rods, how many acres? Ans. 129A. 2R. 25rods.

8. In 6 parcels of wood, each containing 5 cords, 96 feet, how many cords? Ans. 34½cords.

9. How much brandy in 9 casks of 41galls. 3qts. 1pt. each? Ans. 376galls. 3qts. 1pt.

10. If 1 pound of sugar cost 1s. 1d., what will 4lb. cost? Ans. 4s. 4d.

11. If 1 yard of muslin cost 9s. 4¾d. what is the price of 7 yards? Ans. £3 5s. 9¼d.

12. What will 9cwt. of flour amount to at £1 11s. 5d. per cwt.? Ans. £14 2s. 9d.

13. Sold 10 tons of hay, at £8 12s. 6½d. a ton, what is the amount? Ans. £86 5s. 5d.

14. How much will 66 acres of land come to at £7 9s. 6d. an acre? Ans. £493 7s.

15. What will 32 pounds of cheese cost at 3s. 11d. a pound? Ans. £6 5s. 4d.

16. Bought 63 gallons of wine at 5s. 4d. per gallon; what was the amount? Ans. £16 16s.

17. What is the value of 336 yards of linen at 2s. 5d. per yard? Ans. £40 12s.

18. How much will 240 bushels of wheat come to at 14s. 6d. per bushel? Ans. £174.

19. If one pound of sugar cost 1s. 1½d. what will 109 pounds come to? Ans. £6 2s. 7½d.

20. What will 400 pounds of lead come to at 8½d. per pound? Ans. £14 3s. 4d.

21. How much will 1500 gallons of oil amount to at 6s. 2d. per gallon? Ans. £462 10s.

22. A goldsmith bought 11 ingots of silver, each weighing 4℔. 1oz. 15dwt. 22grs. What is the weight of the whole? Ans. 45℔ 7oz. 15dwt. 2grs.

23. A grocer bought 5 hogsheads of sugar, weighing each 12cwt. 1qr. 27℔. How much did the whole weigh? Ans. 6cwt. 1qr. 23℔.

24. Sold 10 pieces of cloth measuring each 17yds. 3qrs. 2na. How many yards were there in all? Ans. 178yds. 3qrs.

25. There are 5 bags of apples, each of which contains 2 bushels, 3 pecks. How many bushels are there in the whole? Ans. 13bush. 3pks.

DIVISION OF A COMPLEX NUMBER BY A SIMPLE NUMBER.

124. *When the dividend only is complex, and the units of the dividend and divisor are different, we first divide the principal units of the dividend according to the common rule. The remainder of this division is then reduced* (57) *to units of the next lower denomination; and to their sum are added the units of the same name in the dividend. We divide this new dividend as before, and if there is a remainder, reduce it to units of the third kind, and to their*

sum add those of the same name in the dividend. After dividing as before, the remainder is reduced to the next lower kind of units; and this process continued as long as inferior denominations are found in the dividend.

EXAMPLE.

We have £4783 3s. 9d. given for the purchase of 87 acres of ground. What is the price per acre?

```
£4783 3s. 9d. | 87
  433         |---------
   85         | £54 19s. 7d.
 -----
 1703s.
  833
   50
 -----
  609d.
  000
 -----
```

It is necessary to divide £4783 3s. 9d. by 87, commencing with the pounds.

The quotient of £4783 divided by 87 in the usual manner, is 54 pounds, with £85 for remainder. These, reduced to shillings (57), give with the 3s. of the dividend, 1703s.; which, divided by 87, give 19s. for quotient, and 50s. for remainder. This remainder reduced to pence, gives, with the 9d. of the dividend, 609d. which divided by 87, give 7d. for quotient and no remainder.

125. But if the dividend and divisor have units of the same kind, before performing the division, it is necessary to examine whether the quotient ought to be of the same kind with them. The condition of the question will decide this inquiry.

126. In case where the dividend and divisor being of the same kind, the quotient ought to be of the same kind with them, division is performed as in the preceding example. For instance, let this question be proposed. If £1243 have gained £7254, how much has one pound gained? The quotient must evidently contain units of the same kind as the dividend and divisor; it ought, therefore, to be pounds. And when £7254 have been divided by 1243, we reduce the remainder to shillings,

as in the preceding example; the second remainder is also reduced to pence, and we find £5 16s. 8d. and $\frac{760}{1243}$ of a penny, for the answer to the question.

127. But when the dividend and divisor being of the same kind, the quotient is required in units of a different kind; the operation must be commenced (57) by reducing the dividend and the divisor, each to the lowest kind in the dividend. Division is then performed as in the preceding case, and we consider the units of the dividend as if they were of the kind required in the quotient. For example, if this question be proposed; how much land can be purchased for £7954 11s. 7d., at £72 an acre? It is plain, from the nature of the question, that the answer must be in acres and parts of an acre. We therefore reduce 7954 pounds, 11s. 7d. to pence, and have 1909099 pence. In like manner, 72 pounds are reduced to pence, and give 17280 pence. We now divide 1909099, considered as acres, by 17280; and have for quotient 110 acres, 2 roods, 36 rods and $\frac{14560}{17280}$ of a rod.

DIVISION OF A COMPLEX NUMBER BY A COMPLEX NUMBER.

128. *When the divisor is also a complex number, it must be reduced to the lowest denomination expressed in it (57,) in order to multiply the dividend by the number, which expresses how many parts of the lowest kind in the divisor, are necessary to compose the principal unit in this divisor. The division is then performed as in the preceding case where the divisor was a simple number.*

EXAMPLE.

If £854 17s. 11d. have been paid for 57F. 5ft. 5in. of work, how much did it cost per fathom?

It is necessary to divide £854 17s. 11d. by 57F. 5ft. 5in. For this purpose we reduce 57F. 5ft. 5in. to inches, which gives 4169 for the new divisor; and as 72 inches make 1 fathom, which is the principal unit of the divisor, we multiply the proposed dividend £854 17s.

11d. by 72 (121), which gives £61552 10s. for the new dividend, and divide as follows.

£61552 10s.	4169
19862	
3186	£14 15s. $3\frac{1833}{4169}$d.
63730s.	
2 040	
1195	
14340d.	
1833	

£61552 divided by 4169 give £14 for quotient, and 3186 for remainder. These £3186 reduced to shillings, give, with the 10s. of the dividend, 63730s., which, divided by 4169, give 15s. for quotient, and 1195s. for remainder. This remainder, reduced to pence, gives 14340 pence, which, divided by 4169, give 3d. for quotient, and 1833 pence for remainder. The quotient, therefore, is £14 15s. $3\frac{1863}{4167}$d.

To understand the reason of this rule, we must observe, that 57F. 5ft. 5in. have the value of 4169 inches; and the inch being the seventy second part of the fathom, the divisor is $\frac{4169}{72}$ of a fathom. But, to divide by a fraction, the fraction which is the divisor (109) must be inverted; the dividend is then multiplied by the divisor thus inverted. We must, therefore, multiply by $\frac{72}{4169}$; which is the same thing as to multiply at first by 72, and then divide by 4169, according to the rule which has been given.

As division by a complex number has, as we have seen, been reduced to division by a simple number, the same attention is required to the nature of the units, as in articles (126) and (127).

The multiplication and division of complex numbers reciprocally prove each other, as in simple numbers.

EXAMPLES FOR PRACTICE.

1. Divide £8 equally among 8 persons; What does each receive? Ans. £1.

2. If 2lb. 9oz. 5dwt. 12grs. of silver be wrought into a dozen spoons, what will each weigh?

Ans. 2oz. 15dwt. 11grs.

3. Divide 17cwt. 1qr. of sugar equally among 6 persons; what does each receive?

Ans. 2cwt. 3qrs. 14lb.

4. If 105Els. Eng. will make 12 suits of clothes, how much does it take for one suit?

Ans. 8Els. Eng. 3qr. 3na.

5. If a man travel 217mi. 5fur. 19pol. 12ft. 6in. in 7 days, how far does he go in one day?

Ans. 31mi. 31pol. 6ft. 6in.

6. If 9 fields of equal extent contain 129ac. 2roods. 25rods, what is the measure of each?

Ans. 14acr. 1R. 25pl.

7. In 6 equal parcels of wood there are 34½ cords, how much is in [illegible] parcel? Ans. 5 cords, 96ft.

8. If 26[illegible]hels of wheat cost $287 97cts. 3mills, what is it per bushel? Ans. $1 9cts. 4mills and rem.

9. Sold 3 yards of muslin for £3 9s. 6d.; what was the price per yard? Ans. £1 3s. 2d.

10. Paid 17s. 6d. for 4 bushels of salt; how much was it per bushel? Ans. 4s. 4½d.

11. If 8 pounds of sugar are sold for 10s. 6d. what is the price per pound? Ans. 1s. 3¾d.

12. Bought [illegible] yards of linen for £3 11s. 8d., what was the price per yard? Ans. 8s. 11½d.

13. Sold 132 yards of cloth for £221 18s. 6d. How much was it per yard. Ans. £1 13s. 7½d.

14. What is the price of a bushel of wheat, when 42 bushels are sold for £17 13s. 6d. Ans. 8s. 5d.

15. Bought two pieces of linen, one of which contained 30 yards, and the other 25 yards; the price was 7s. 6d. per yard; what was the cost of the two pieces?

Ans. £20 12s. 6d.

16. A farmer has three tracts of land, the first contains 125 acres, 3 roods; the second 200 acres, 2 roods. 18 rods; the third 175 acres, 10 rods. He intends to divide this land equally between his two sons; what will be the share of each son? Ans. 250acr. 2roods. 14pol.

17. Three merchants A, B, and C, own a ship in company; A owns $\frac{5}{8}$, B, $\frac{2}{8}$, and C, $\frac{1}{8}$; they have receiv-

ed for freight £228 16s. 8d. It is required to divide it among the owners according to their respective shares; what is the share of each? Ans. A's £143 5d. B's £57 4s. 2d. C's £28 12s. 1d.

DUODECIMALS.

The best system of Numeration, perhaps, is that, which, without employing a great number of characters, contains in the scale of its gradations the greatest number of divisors. In this respect the duodecimal system deserves the preference. If two characters had been added to the decimal system, it would be much more convenient; as it would express, by means of the divisions of this system, the third and the fourth part of the principal unit. The advantages of such a method of numeration are apparent in the duodecimal divisions of linear measure. Thus, the foot is divided into twelve inches, the inches into twelve lines or *primes*, the primes into twelve parts called *seconds*, these last into twelve parts called *thirds;* and so on.

Inches are sometimes called primes and marked thus ('). The divisions smaller than primes are then successively marked as follows; (''), ('''), (''''), &c.

For example, if we wish to express four feet, six primes, two seconds, and one third, we write 4f. 6' 2'' 1'''.

The following is the rule for the multiplication of this kind of numbers.

Place the several terms of the multiplier under the corresponding denominations of the multiplicand. Multiply the whole multiplicand, beginning at the lowest denomination in it, by the highest denomination in the multiplier, observing to carry an unit for every 12, in passing from a lower to a higher denomination. In like manner, multiply the whole multiplicand by the next lower denomination in the multiplier, and place the result one remove towards the right Perform the same operation with the remaining denominations of the multiplier; add the several products thus found, and they will form the product required.

EXAMPLE.

Multiply	2f.	4′	
by	3	10′	
	7	0	
	1	11	4″
	8	11′	4″ Answer.

In this operation, the 4 inches, or primes, may be considered in reference to the denomination of feet, as 4 twelfths, or $\frac{4}{12}$; we therefore multiply $\frac{4}{12}$ by 3, and have $\frac{12}{12}$ or 1 for the product, which added to the product of 2 by 3 gives 7 feet and 0 primes. We next multiply 4′ or $\frac{4}{12}$ by 10, considered as $\frac{10}{12}$, the product of which is $\frac{40}{144}$, or $\frac{40}{12}$ of $\frac{1}{12}$, or 40″, which reduced gives 3′ 4″; putting down the 4″ one remove ~~towards the~~ right, we reserve the 3′ to be added to the product of 2 feet by 10′, or $\frac{10}{12}$, which product is $\frac{20}{12}$ of a foot, to which, 3 being added, we have $\frac{23}{12}$ft. or 1ft. and 11″. Taking the sum of these results, we have 8ft. 11′ 4″, for the product of 2ft. 4′ by 3ft. 10′. This method of operation may be extended to cases, in which there are a greater number of denominations.

EXAMPLES FOR PRACTICE.

1. Multiply 4ft. 7′ 8″ by 9ft. 6′.
Ans. 44ft. 0′ 10″.

2. Multiply 7ft. 8′ 6″ by 10ft. 4′ 5″
Ans. 79ft. 11′ 0″ 6‴ 6⁗.

3. What is the price of a marble slab, whose length is 5ft. 7′, and breadth 1ft. 10′, at one dollar per foot?
Ans. $10 23cts.

4. There is a house with three tiers of windows, and three windows in a tier; the height of the first tier is 7ft. 10′, of the second, 6ft. 8′, and of the third, 5ft. 4 ; and the breadth of each is 3ft. 11′; what will the glazing come to at 14d. per foot? Ans. £13 11s. 10½d.

5. How many cubic feet of wood in a load 7ft. 10′ long, 3ft. 11″ wide, and 3ft. 6″ high?
Ans. 107ft. 4′ 7′

6. The solid contents of a wall is required, which is 53ft. 6′ long, 10ft, 3′ high, and 2ft. thick.

Ans. 1096ft. 9′.

A number written in the decimal system may be transferred into the duodecimal system, by observing the following rule. *Divide the proposed number by* 12, *and write the remainder, or cipher if there is no remainder. Divide the first quotient by* 12, *and write the remainder to the left of the first remainder. Divide the second quotient by* 12, *and write the remainder at the left of the second remainder; and so on. And by continuing the operation, you will at length write in the duodecimal system, the number first written in the decimal system.*

FORMATION OF SQUARE NUMBERS AND THE EXTRACTION OF THEIR ROOTS.

129. The product, which results from the multiplication of any number by itself, is called *the square* of that number. Thus, 25 is the square of 5, because it is the product of 5 multiplied by 5.

130. *The square root* of any number, is that number which, multiplied by itself, produces that same number. Thus, 5 is the square root of 25; 7 is the square root of 49.

131. A *square number* is that, which has been made both multiplicand and multiplier; and consequently, is twice a factor of the product (42); hence, this product, or square, is also called the *second power* of that number.

No rule is necessary for squaring a number, but to multiply that number by itself, according to the common rules of multiplication. But for extracting the square root of a number, that is, for descending from the square to the root, a rule is necessary; particularly when the number, or square proposed consists of more than two figures.

When the given number is expressed by only one or two figures, its root, if a whole number, is expressed by some one of the following numbers.

1, 2, 3, 4, 5, 6, 7, 8, 9.

Of which the squares are

1, 4, 9, 16, 25, 36, 49, 64, 81.

Thus the square root of 72, for example, is 8, when expressed in whole numbers; because 72 being between 64 and 81, its root is a number contained between 8 and 9, the roots of those numbers; it is, then, 8 and a fraction. The value of this fraction cannot be exactly determined, but approximations to it may be made continually, as will be seen hereafter.

132. The square root of a number, which is not a perfect square, is called a *surd, irrational, or incommensurable number.*

133. We will now consider numbers, which are expressed by more than two figures. The method, which must be followed for finding the root of a number, is deduced from observations made upon what takes place in forming the square. For example, to square the number 54;

$$\begin{array}{r} 54 \\ 54 \\ \hline 216 \\ 270 \\ \hline 2916 \end{array}$$

After writing the multiplicand and multiplier in the usual manner, we multiply 4 in the multiplicand by 4 in the multiplier, and the product is evidently the *square of the units.* We next multiply 5 in the multiplicand by 4 in the multiplier, which gives *the product of tens by units.* Passing now to the second figure of the multiplier, we multiply 4 in the multiplicand by 5 in the multiplier, which gives (44) *a product of tens by units.* Lastly, we multiply the upper 5 by the lower 5, and the product is *the square of the tens.* Adding these products, we have for the square, the number 2916, which, as has been seen, is composed of the *square of the tens, plus twice the product of the tens by the units, plus the square of the units* of the number 54.

134. What has been observed, being an immediate consequence of the rules of multiplication, has the same application to any other number composed of tens and units, as to 54; *hence, in general, the square of every number composed of tens and units, contains these three*

parts, which have been mentioned, namely, the square of the tens of the number, twice the product of the tens by the units, and the square of the units.

135. As the square of the tens is hundreds, (since 10 times 10 make 100), it is plain, that the square of the tens cannot make a part of the two last figures of the whole square. In like manner, the product of double the tens multiplied by the units being necessarily tens, cannot make a part of the last figure of the whole square.

136. Then, to descend from the square 2916 to its root, we must reason in the following manner.

EXAMPLE I.

```
2916 | 54
416
104
----
000
```

We commence the operation by finding the tens of the root; but the formation of the square teaches us, that there is, in 2916, the square of the tens, and that this square cannot make a part of its two last figures; it is then contained in 29; and as the square root of 29 cannot be more than 5, we conclude that the number of tens in the root is 5, we therefore place 5 by the side of 2916, as in the example. We next square 5 and subtract the product 25, from 29; there remains 4, by the side of which are brought down 16, the two remaining figures of the given number 2916. Now to find the units of the root, we must direct our attention to what is contained in the remainder 416. It can only contain two parts of the square, namely, the double of the tens of the root, multiplied by the units, and the square of the units of the same root. Of these two parts we need only use the first, to find the units which are sought; because, since this part is formed of double the tens multiplied by the units, if we divide it by double the tens, which are known, the quotient will be the units (74). Nothing now remains but to find in what part of 416, the double of the tens, multiplied by the units, is contained. But we have remarked above, that it cannot make a part of the last figure; it is then contained in 41. Hence, it is necessary to divide 41 by 10, the double of the tens already found. We now write under 41, the double of the tens already found, or 10; and performing division, 4 the quotient, which results, is the number of units; which we place to the right of the 5 tens; hence the root sought is 54.

It must be observed that, although the quotient thus found may be the true root, yet it sometimes happens that the quotient found in this manner may be greater than the true one; because 41 (that is, the part which remains after the separation of the last figure,) contains not only the double of the tens multiplied by the units, but also the tens arising from the square of the units. Wherefore, to prevent all doubt concerning the figure in the place of units, the following verification must be made.

After having found 4, the figure of units, and written it in the root, we place by the side of it 10, the double of the tens, which makes 104; each figure of which is successively multiplied by the same number 4, and the successive products are subtracted from the corresponding parts of 416; as there is no remainder, we conclude that 54 is the true root.

If there is a remainder, the root may, nevertheless, be the true root in whole numbers; at least, if the remainder is not greater than the double of the root augmented by unity.

The verification, which has been explained, depends upon the formation of the square; for when 104 is multiplied by 4, it is plain, that we form the square of the units and the double of the tens multiplied by the units; that is, the perfect square is completed.

137. *From what has been said it may be concluded, that, to extract the square root of a number expressed by four figures, it is necessary, after having separated two figures upon the right, to seek the square root of the two figures, which remain on the left. This root will be the tens of the whole root sought, and is written by the side of the given number with a line between them. We then subtract from the left hand division, or the two left hand figures, the square of the root which has been found;*

and after having written the remainder under this division, we bring down to the side of the remainder, the two figures which were separated. We next separate the unit figure of the division which has been brought down, and divide the figures which remain on the left, by the double of the tens, which are written underneath. The quotient is written by the side of the first figure of the root, and afterwards by the side of double the tens which served for the divisor. Lastly, we multiply by the same quotient, all the figures in this last line, and subtract their product from corresponding figures in the line above.

This operation may be explained by the following example.

EXAMPLE II.

The square root is required of 7569.

75 69 | 87 root.
116·9
167
———
000

We separate the two figures 69, and seek the square root of 75; it is 8. We write 8 by the side of the given sum, drawing a line between them, and square the number 8; from 75 we subtract 64, the square of 8, and 11 remains. By the side of 11, we bring down 69, the division which was first separated. In 1169, the last figure 9 is separated, in order to know the part, 116, which must be divided to find the units.

We next form our divisor by doubling the 8 tens which have been found, and write the divisor under 116; the quotient of this division is 7, which is written in the root by the side of 8, in the place of units. We carry this quotient also by the side of the divisor 16, and multiply 167, which forms the last line, by the same quotient 7, and subtract the products, as they are found, from 1169. There is no remainder, which proves, that 7569 is a perfect square, and the square of 87.

138. We may remark, that only the part, which remains at the left after having separated the last figure, need be divided by twice the tens; because, if this part

does not contain twice the tens, as the separated figure will afford no assistance in the following operation, 0 must be put in the root. If on the contrary, we find that twice the tens is contained in this part more than 9 times, only 9 must be placed in the root. The reason for this is the same as that given for division in article (66.)

139. After having well understood what has been said concerning the square root of numbers expressed in four figures, we shall find no difficulty in understanding the method, which must be followed when the number of figures is larger. *Of whatever number of figures the root is composed, we may always conceive it to be composed of two parts, one of which may be called tens, and the other, units.* For example, 874 can be considered as representing 87 tens and 4 units. Now, when the two first figures of the root have been found by the preceding method, the third also may be found by the same rule, by considering the two first figures as forming only a single number of tens, to which the same reasoning is applied, to find the third, as was applied to the first, for finding the second. In like manner, when we have found the three first figures, if a fourth is required, the three first are considered as forming only a single number of tens, to which we apply the same reasoning for finding the fourth, as to the two first, for finding the third; and so on.

But to proceed with order, *the given number is separated into divisions of two figures each, passing from right to left; the last division, however, need contain but one figure.* The reason for this preparation is founded on the principle that, considering the root as composed of tens and units, according to what has been said in article (135 and following), it is necessary to commence by separating the two last figures upon the right, in order to have in the part which remains on the left, the square of the tens. But as this part is also composed of more than two figures, similar reasoning directs to the separation of two figures again upon the right; and so on.

The following example will explain this operation.

EXAMPLE III.

The square root is required of 76807696.

```
76 80 76 96)8764
128·0
 167
------
 1117·6
  1746
-------
  7009·6
  17524
-------
  00000
```

After having separated the given number into divisions of two figures each, passing from right to left, we seek the square root of the division 76, which is first on the left, and find it to be 8; 8 is therefore written by the side of the given number. We next square 8, and subtract 64, its square, from 76, and have 12 for remainder, which is written under 76. By the side of this remainder we bring down 80, the next division, the last figure of which is separated by a point; and below the part 128, we write 16, twice the root found, and then say, in 128 how many times 16? It is contained 7 times; and 7 is written in the root, at the right of 8, and also by the side of 16, twice the root already found. We then multiply 167 by the same number 7, and subtract the product of this multiplication from 1280; there remains 111, by the side of which we bring down 76, the next division, which gives 11176. We separate 6, the last figure of this number, and under 1117, the part which remains to the left, write 174, twice the root 87; we divide 1117 by 174, and having found 6 for the quotient, write 6 in the root and by the side of 174, twice the root 87; we multiply 1746 by the same number 6, and subtract the product from 11176; there remains 700, by the side of which the division 96 is brought down, and 6, the last figure of it, separated by a period. Below 7009, which remains at the left, we write 1752, twice the root 876; and dividing 7009 by 1752, find 4 for quotient, which is written in the root, and by the side of 1752. We multiply 17524 by the same number 4, and subtract the product from 70096, and nothing remains; thus the square root of 76807696 is exactly 8764.

140. When the given number is not a perfect square, there is a remainder at the end of the operation, and the square root which has been found, is the square root of a greater square contained in the given number. In this case, it is not possible to extract the square root exactly; but approximations may be made to the true root, in such manner, that the error, which results in the square, shall be less than a given quantity.

This approximation is conveniently made by means of decimals. *To the right of the given number, place twice as many ciphers as there are decimal figures required in the root; perform the operation in the usual manner, and afterwards separate by a comma upon the right of the root, half as many figures for decimals as there were ciphers joined to the given number.* As the product of two numbers (54) containing decimals, ought to contain as many decimals as there are in both the factors, the square (of which the two factors are equal,) ought to contain twice as many decimals as there are in one of the factors, that is, twice as many as are required in the root.

EXAMPLE.

The square root of 87567 is required to three places of decimals. To express thousandths, or the required decimal root, three figures are necessary. Six ciphers, therefore, must be placed to the right of the square 87567; hence, the square root of 87567000000, must be extracted.

$$\begin{array}{l|l}
8\cdot75\cdot67\cdot00\cdot00\cdot00 & 295917 \\
47\cdot5 & \\
\ \ 49 & \\
\hline
\ \ 346\cdot7 & \\
\ \ \ \ 58\,5 & \\
\hline
\ \ \ \ 5420\cdot0 & \\
\ \ \ \ \ \ 590\,9 & \\
\hline
\ \ \ \ \ \ 10190\cdot0 & \\
\ \ \ \ \ \ \ \ 5918\,1 & \\
\hline
\ \ \ \ \ \ \ \ 427190\cdot0 & \\
\ \ \ \ \ \ \ \ \ \ 59182\,7 & \\
\hline
\ \ \ \ \ \ \ \ \ \ 129111 &
\end{array}$$

After performing the operation as in the preceding example, we find, for the square root to the third place of decimals, the number 295917. This root is the root of 87567000000; but as the root of 87567,000000, or of 87567, only is required; we separate half as many figures for decimals in the root, as ciphers were added to the square; which gives 295,917 for the square root of 87567, to three places of decimals.

In like manner, if the square root of 2 is required to four places of decimals, we extract the square root of 200000000, which is 14142; separating the four figures at the right by a comma, we have 1,4142 for the square root of 2, found to four places of decimals.

141. It has been seen in article (106,) that when one fraction is to be multiplied, by another, the numerator must be multiplied by the numerator, and the denominator by the denominator; consequently, *when a fraction is to be squared, it will be necessary to square the numerator and the denominator.* Thus, the square of $\frac{2}{3}$ is $\frac{4}{9}$, that of $\frac{4}{5}$ is $\frac{16}{25}$.

142. Then reciprocally, *to extract the square root of a fraction, it will be necessary to extract the square root of the numerator and also that of the denominator.* Thus, the square root of $\frac{9}{16}$ is $\frac{3}{4}$, because that of 9 is 3, and that of 16, is 4.

143. It sometimes happens, that the numerator, or the denominator, or both, are not perfect squares. *If the numerator only is not a square, the approximate root is extracted by the method already explained; and having extracted the root of the denominator, we give to it for numerator, the root of the numerator.* Thus, if the square root of $\frac{2}{9}$ is required, we extract the approximate root of the numerator 2, which is 1,4, or 1,41, or 1,414, or 1,4142, &c., according as the root is required to a greater or less degree of exactness. And as the square root of 9 is 3, we have for the approximate root of $\frac{2}{9}$, the quantity $\frac{1,4}{3}$ or $\frac{1,41}{3}$ or $\frac{1,414}{3}$ or $\frac{1,4142}{3}$. *But if the denominator is not a square, both terms of the fraction are multiplied by the same denominator; (which does not change the value) the denominator is thus made a square,* and the operation performed as in the preceding case. For example, if the square root of $\frac{3}{5}$ is required, we change this fraction to $\frac{15}{25}$; extracting the square root of 15, to three places of decimals, for instance, we have 3,872; and as the square root of 25 is 5, the square root $\frac{15}{25}$ will be $\frac{3,872}{5}$.

144. In order to avoid fractions of different kinds in the same sum, we reduce the result $\frac{3,872}{5}$, to decimals, by dividing 3,872 by 5, which gives 0,774 for the root of $\frac{3}{5}$ expressed in pure decimals (99.)

145. *If whole numbers are joined to fractions, the whole numbers are reduced to fractions* (86), *and the operation performed according to the direction given for extracting the roots of fractions.* Thus, to extract the square root of $8\frac{3}{7}$, we change $8\frac{3}{7}$ into $\frac{59}{7}$, and this again into $\frac{413}{49}$ (143); of which the approximate root is $\frac{20,322}{7}$, or 2,903.

146. *The fraction, which accompanies the whole number, may also be reduced to decimals;* but it is necessary to observe, that the number of decimals employed must be even, and twice as great as the number of decimal figures required in the root. This preparation is made by placing after the given sum 1, 3, or 5, &c. ciphers, which does not change its value (30). Thus, to extract the square root of 21,935 to three places of decimals,

we extract the square root of 21,935000, which is 4,683; this is also the root of 21,935. In like manner, the square root of 0,542, expressed in three decimal figures, is 0,736; and that of 0,0054, to the same value 0,073.

147. When, by the method already explained, we have found the three first figures of the root, many others may be found, with greater facility, by division only, in the following manner. Take, for example, the number 763703556823; we commence by seeking the three first figures of the root as in the preceding manner, and find 873 for the root, and 1574 for remainder. By the side of this remainder we place 55, the two figures, which follow 763703, the part, which gave the three first figures of the root. (If four figures of the [illegible] have been found, three figures are placed by the si[illegible]he remainder; if five figures of the root have been found, the four following figures are joined to the remainder; and so on.) We then divide 157455 by 1746, twice the root found, and have 90 for the quotient: these are two new figures of the root, which has now become 87390. We square this root, and subtract its square 7637012100, from the part 7637035568, of which 87390 is the root; and there remains 23468.

If other figures of the root are required; as we have already obtained five, the four next may be found by division only. For this purpose, we place after the remainder 23468, the remaining figures, 23, of the given number, and two ciphers, and divide 2346832300, by 174780, twice the root already found, and have 1342, for the four new figures of the root. But by separating the given number into divisions, according to the requirement of the rule, it is plain, that the root should contain only six figures of whole numbers; the root is then 873901,342, to three places of decimals.

The abbreviation, which has been explained, is a consequence of the general principle, which it is easy to deduce from what has been said in article (134), *that the square of any quantity composed of two parts, contains the square of the first part, twice the first part multiplied by the second, and the square of the second.*

EXAMPLES FOR PRACTICE.

1. What is the square root of 152399025? Ans. 12345.

2. What is the square root of ,00032754? Ans. ,01809.

3. What is the square root of $\frac{5}{12}$? Ans. ,645497.

4. What is the square root of 6$\frac{2}{5}$? Ans. 2,5298, &c.

5. What is the square root of 10? Ans. 3,162277, &c.

6. What is the square root of 234,09? Ans. 15,3.

7. What is the square root of 1030892198,4001? Ans. 32107,51.

8. A general has an army of 4096 men; how many must he place in rank and file to form them into a square? Ans. 64.

9. A wall is 36 feet high, and a ditch before it is 27 feet wide; what is the length of a ladder, that will reach to the top of the wall from the opposite side of the ditch? Ans. 45 feet.

10. It is required to lay out 25600 square rods of land in a square; what is the side of a square that shall contain the land? Ans. 160 rods.

11. A gentleman purchased 3025 tiles, for the purpose of paving a square; how many tiles will there be upon a side? Ans. 55.

12. A certain general commanded an army of 49284 men; and the better to secure his standard, he gave orders to form into a square body, the men standing 4 feet distant; what number of men will be on a side; and what quantity of land will they occupy? Ans. 222 men occupy 17,28 acres.

13. What length of rope must be tied to a horse's neck, that he may graze upon 7854 square feet of new feed every day, for 4 days; the end of the rope being each day fastened to the same stake? To perform this and similar questions, *divide the given area by ,7854 and the square root of the quotient will be the diameter of the circle fed over by the horse; half of this diameter will be the length of the rope required.* Hence, the first rope will be 50 feet, the second 70,5, the third 86,5 and the fourth 100 feet in length.

14. Four men, A, B, C, and D, purchased a large grindstone, the diameter of which was 200 inches; they agreed that D should wear off his share first, and that each man should have it alternately till they had worn off their shares; how much must each man wear off round the stone?

Ans. D. $13\frac{1}{2}$; C. 16; B. $20\frac{1}{2}$; A. 50 inches.

NOTE.—*The diameter of a circle being given, the area is found by squaring the diameter, and multiplying its square by ,7854.*

15. What is the square root of 106929? Ans. 327.
16. What is the square root of 36372961?
Ans. 6031
17. What is the square root of 3271,4007?
Ans. 57,19, &c.
18. What is the square root of 10,4976? Ans. 3,24.
19. What is the square root of $\frac{7056}{9216}$? Ans. $\frac{7}{8}$.
20. What is the square root of $\frac{2704}{4225}$? Ans. $\frac{4}{5}$.
21. What is the square root of $37\frac{36}{49}$? Ans. $6\frac{1}{7}$.
22. What is the square root of $27\frac{9}{16}$? Ans. $5\frac{1}{4}$.

23. If 484 trees are planted at an equal distance from each other, so as to form a square orchard, how many will be in a row each way? Ans. 22.

24. A certain number of men gave 30s. 1d. for a charitable purpose; each man gave as many pence as there were men: how many men were there? Ans. 19.

25. There is a circle whose diameter is 4 inches, what is the diameter of a circle 4 times as large?

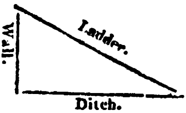
Ans. 8 inches.

NOTE.—*Square the given diameter, multiply this square by the given proportion, and the square root of the product will be the diameter required. If the required be less than the given diameter, divide the square of the given diameter by the proportion.*

26. There are two circular ponds in a gentleman's pleasure grounds; the diameter of the less is 100 feet, and the greater is three times as large. What is its diameter? Ans. 173,2 &c.

27. If the diameter of a circle be 12 inches, what will be the diameter of another circle, half as large?
Ans. 8,48, &c. inches.

28. A certain castle, which is 45 yards high, is surrounded by a ditch 60 yards broad; what must be the length of a ladder to reach from the outside of the ditch to the top of the castle? Ans. 75 yards.



NOTE. This diagram is called a right angled triangle; *and the square of the hypothenuse, or longest side, is equal to the sum of the squares of the two other sides. Now if the square root of the sum of the squares of the height of the wall and breadth of the ditch be extracted, we shall have the length of the ladder. If the difference of the squares of the longest and either of the other sides be taken, we shall have the square of the remaining side; the square root of this square will be the length of the side.*

29. A line 27 yards long will exactly reach from the top of a fort to the opposite bank of a river, which is known to be 23 yards broad; what is the height of the fort? Ans. 14;142, &c. yards.

30. Suppose a ladder 40 feet long, be so planted as to reach a window 33 feet from the ground, on one side of the street, and without moving it at the foot, will reach a window on the other side 21 feet high; what is the breadth of the street? Ans. 56,60, &c. feet.

FORMATION OF CUBE NUMBERS, AND THE EXTRACTION OF THEIR ROOTS.

140 What is called the *cube* of a number, is formed by multiplying the number by itself, and afterwards multiplying the product of this multiplication by that same number.

The cube of a number, properly speaking, is the product of the square of a number, multiplied by that number. Thus, the cube of 3 is 27, because it is what results from the multiplication of 9, the square of 3, by 3, The number which has been cubed, is then three times factor in the cube; and for this reason, the cube is also called the *third power* of the number.

150. In general a number is said to be raised to the second, third, fourth, fifth, &c. power when it has been multiplied 1, 2, 3, 4, &c. times, consecutively, by itself; or, when it is 2, 3, 4, 5, &c. times factor in the product.

151. The *cube root* of a number, is the number which, multiplied by its square, produces the cube. Thus, 3 is the cube root of 27.

152. There is no need of rules for forming the cube of a number; but, to descend from the cube to its root, a method is necessary.

We may observe, however, that rules are not necessary for extracting the cube root in whole numbers, when the given number is expressed by four, or less than four figures; because, 1000 being the cube of 10, every number below 1000, and consequently of less than four figures, will have for its root less than 10, that is, less than two figures. Thus every number, which intervenes between two of the following, namely 1, 8, 27, 64, 125, 216, 343, 512, 729, will have its cube root, in whole numbers, between two numbers corresponding to this series, 1 2 3 4 5 6 7 8 9, of which the first contain the cubes.

153. *Every number has not a cube root;* but we can make continual approximations to a number, which, being cubed, will continually approximate to the reproduction of that first number. This will be shown, when we have explained the method for finding the root of a perfect cube.

154. We will now examine of what parts the cube of a number must be composed, which shall contain tens and units. Since the cube results from the square of a number multiplied by that same number, it is essential to recollect (134,) that *the square of a number composed of tens and units, contains,* 1st, *the square of the tens,* 2d, *twice the product of the tens by the units,* 3d, *the square of the units.* To form the cube then, it is necessary to multiply these three parts, by the tens and by the units of the same number. That we may the

more clearly understand what products result from this operation, we will give to it the following form.

1ST.

The square of the tens,
Twice the product of the tens by the units,
The square of the units,

being multiplied by the tens, will give

The cube of the tens.
Twice the product of the tens multiplied by the units.
The product of the tens by the square of the units

2D.

The square of the tens,
Twice the product of the tens by the units,
The square of the units,

being multiplied by the units, will give

The product of the square of the tens multiplied by the units.
Twice the products of the tens by the square of the units.
The cube of the units.

Then taking these 6 results, and adding such as are similar, we perceive, that the cube of a number composed of tens and units, contains four parts, namely, *the cube of the tens, three times the square of the tens multiplied by the units, three times the tens multiplied by the square of the units, and the cube of the units.*

After this manner, form the cube of a number composed of tens and units, of 43, for example.

$$\begin{array}{r}64000\\14400\\1080\\27\\\hline 79507\end{array}$$

First take the cube of 4, which is 64; but as this 4 is tens, its cube will be thousands; because the cube of 10 is a thousand; hence the cube of 4 tens will be 64000.

3 times 16, or three times the square of 4 tens, being multiplied by the 3 units, will give 144 hundreds, because the square of 10 is 100; this product will then be 14400.

3 times 4, or 3 times the tens, being multiplied by the square of the units, or 9, will give tens, and this product

will be 1080. Lastly, the cube of the units, which is 27, will be confined to the place of units.

Adding these parts, we have 79507 for the cube of 43. This cube might have been more readily found by multiplying 43 by 43, and then multiplying 1849, the product, again by 43. But it has not been so much our object to find the cube, as to understand, by an examination of the parts which compose it, the method of descending to its root.

155. After this explanation, we proceed to the extraction of the cube root.

EXAMPLE.

The cube root of 79507 is required. For reasons which will hereafter be explained, when the cube root of a number is required, *the number must be separated into divisions of three figures each.*

Cube.	Root.
79·507	43
155·07	
48	

To obtain the part of this number, which contains the cube of the tens in the root, we separate the three last figures of it, in which it has been seen, that the cube cannot be contained, since it has the value of thousands. We seek the cube root of 79; it is 4, which is written in the place of the root. We cube 4, and subtract the product 64, from 79; there remains 15, which is written under 79. By the side of 15, we bring down 507, which gives 15507, in which ought to be contained 3 times the square of the 4 tens already found, multiplied by the units sought, plus three times the same tens multiplied by the square of the units, plus the cube of the units. We separate the two last figures 07; the part 155, which remains at the left, contains 3 times the square of the tens multiplied by the units; therefore, to obtain the units (74), we divide 155 by 3 times the square of the 4 tens, that is, by 48. We find that 48 is contained in 155, 3 times; 3 is then written in the root. To prove this root and find the remainder, if there is one, we add together the three parts of the cube, which ought to be found in 15507, in order to know if they form 15507, or by how much they differ from it. It is also convenient to make this verification by cubing the number 43; that is, by multiplying 43 by 43, and then multiplying 1849, their product, by 43, which gives 79507 for product. Thus 43 is exactly the cube root. If the given number consist of more than six figures, the operation is conducted as in the following example.

EXAMPLE.

The cube root of 596947688 is required.

```
 596·947·688 | 842
  849·47
   192
 592704
 ---------
 42436·88
  21168
596947688
 ---------
000000000
```

We consider the root as composed of tens and units, and therefore commence by separating the three last figures. The part 596947, which contains the cube of the tens, consisting of more than three figures, will have its root expressed by more than one figure; consequently, by tens and units. It is necessary then in order to find the cube of the first tens, to separate the three figures 947.

This done, we seek the cube root of 596; it is 8, which is written in the place of the root. We cube 8, and subtract the product 512, from 596; there remains 84, which is written under 596. By the side of 84, we bring down 947, which gives 84947, of which we separate the two last figures. Under the part 849, we write 192, which is 3 times the square of the root 8, and divide 849 by 192; we find 4 for quotient, which is written in the root. To verify this root and find the remainder, we cube 84, and subtract the product 592704, from the number 596947, and have 4243 for remainder. By the side of this remainder we bring down the division 688, and considering the root 84 as a single number which denotes the tens in the root sought, we separate the two last figures 88, from the division brought down, and divide the part 42436 by 3 times the square of 84, that is, by 21168, and find 2 for quotient, which is written to the right of 84.

To verify the root 842 and find the remainder, if there is one, we cube 842, and subtract the product 596947688, from the given number 596947688. There is no remainder, we therefore conclude, that 842 is the exact root of 596947688.

We may observe 1st, that in the course of these operations, more than 9 ought never to be put in the root. 2d, If the figure placed in the root is too large, subtraction cannot be performed; hence, the root must be successively diminished by 1, 2, 3, &c. units, until subtraction can be performed.

When the given number is not a perfect cube, we can only obtain the approximate root; and it rarely happens, that the root is found in whole numbers. Decimals are of great use in extending this approximation to any required degree of exactness, although the exact root may not be found.

156. *When the cube root of an imperfect cube is required in a given number of decimal figures, three times as many ciphers must be placed to the right of it, as there are decimals required in the root. The root is then extracted as in the preceding examples; and after the operation, as many figures are separated on the right as there are decimals required.*

EXAMPLE.

The cube root of 8755 is required to two places of decimals. To obtain hundredths in the root, that is, two decimal figures, it is necessary that the given number should contain six (54); six ciphers are therefore put after 8755. Thus the question is to extract the cube root of 8755000000.

8·755·000·000 | 2061
07·55
12
8000
7550·00
1200
87418 16
131840·00
127308
87545529 81
4470 19

For reasons already explained, this number is separated into divisions of three figures each, passing from right to left. The cube root of the last division 8 is extracted; it is 2, which is written in the root. We cube 2 and subtract the product from 8, and 0 remains. By the side of 0, we bring down the period 755, of which we separate the two last figures 55. Under the remaining part 7, we write 12, three times the square of the root; and dividing 7 by 12, find 0 for quotient, which is written in the root. We cube the root 20, which gives 8000 and subtract it from 8755, and have 755 for remainder. By the side of this remainder we bring down the division 000, and separate the two figures on the right; under the remaining part 7550 we write 1200, which is 3 times the square of the root 20; and dividing 7550 by 1200, find 6 for quotient, which is written in the root. We cube the root 206 and subtract the product from 8755000 and have 13184 for remainder, by the side of which we bring down the last division 000, and separate the two last figures of it. Under 131840, the remaining part, we write 127308, three times the square of 206, the root found. We divide 131840 by 127308, and find 1 for quotient, which is written to the right of 206. We cube 2061, and having subtracted 8754552981, the product, from 8755000000, have 447019 for remainder. The approximate cube root of 8755000000 is 2061; then that of 8755,000000, is 20,61; because the cube contains 3 times as many decimals as its root (54).

If we wish to extend the approximation still further, three ciphers are put after the remainder, and the same operation is performed as at each time when a division is brought down.

157. Since, when a fraction is multiplied by a fraction, the numerator is multiplied by the numerator, and the denominator by the denominator; therefore, *when a fraction is to be cubed, the numerator and the denominator must each be cubed. And reciprocally, when the cube root of a fraction is to be extracted, we must extract the cube root of the numerator and the cube root of the denominator.* Thus, the cube root of $\frac{27}{64}$ is $\frac{3}{4}$, because the cube root of 27 is 3, and that of 64, is 4.

158. *But if the denominator only is a cube, we extract the approximate root of the numerator, and give to this root for denominator, the cube root of the denominator.* For example, if the cube root of $\frac{143}{343}$ is required, as the numerator is not a cube, we extract its approximate root, which to two places of decimals, is 5,22; and extracting the cube root of 343, which is 7, we have $\frac{5,22}{7}$ for the approximate cube root of $\frac{143}{343}$; or by reducing this fraction to decimals (99), we have 0,74 for the root approximated to two places of decimals.

159. *If the denominator is not a cube, both terms of the fraction are multiplied by the square of the denominator; the new denominator being now a cube, the operation is conducted as before.* For example, if the cube root of $\frac{3}{7}$ is required, we multiply the numerator and denominator by 49 the square of the denominator; the fraction is now $\frac{147}{343}$, which has the same value as $\frac{3}{7}$ (88). The cube root of $\frac{147}{343}$ is $\frac{5,27}{7}$; or, by reducing it to decimals, 0,75; the cube root of $\frac{3}{7}$ is then 0,75, approximated to two decimal figures.

If whole numbers are joined to fractions, they are converted into a fraction; the question is then to extract the cube root of a fraction (157 and following).

The given fraction may also be reduced to decimals, whether it be accompanied by whole numbers or not; but care must be taken to extend the reduction to three times as many decimal figures as are required in the root. Thus, if the cube root of $7\frac{3}{11}$ is required in three places of decimals, we change the fraction $\frac{3}{11}$ into 0,272727272; hence, to obtain the cube root of $7\frac{3}{11}$, we extract the cube root of 7,272727272, which is 1,937.

160. *When the cube root is to be extracted from a number, which contains decimals, it must be prepared by placing after it such a number of ciphers as shall make the number of decimal figures either* 3, 6, *or* 9, *&c. The cube root is then extracted in the same manner as if no decimals were in the given sum; but after the operation has been performed, we separate, on the right of the root, as many figures for decimals as shall be equal to one third of the number of decimal figures in the prepared sum. If the root found does not contain so many figures as the rule requires, the deficiency must be supplied by placing ciphers to the left of the root.* Thus, to extract the cube root of 6,54 to three places of decimals, we place after it 7 ciphers, and extract the cube root of 6540000000, which is 1870. To the right of this root we separate 3 figures, because there were 9 decimals in the cube, and have 1,870, or only 1,87 for the cube root of 6,54. In like manner, we find that the cube root of 0,0006, approximated to two places of decimals, is 0,08.

161. When the four first figures of the cube root have been found by the rule already explained, others may be obtained with greater facility by division, in the following manner. If the cube root of 5264627832723456 is required, we seek the four first figures of the root in the usual manner; they are 1739, and 5681413 is the remainder, after the operation. By the side of this remainder we bring down 72, the two figures which follow 5264627832, the part which gave the four first figures of the root. (We bring down the three figures which follow this part when the root found is expressed by five figures, and four, when it is expressed by six). We divide 568141372 by 9072363, three times the square of the root 1739, and have 62 for quotient, which are to be placed to the right of 1739 as two new figures of the root. And 173962 is the cube root of the given sum expressed in whole numbers.

If the operation is to be continued further, we cube this root, and having subtracted the product from the given number, place four ciphers after the remainder, and divide the whole by three times the square of 173962; which gives four decimals for the root.

EXAMPLES FOR PRACTICE.

1. What is the cube root of 1092727? Ans. 103.
2. What is the cube root of 27054036008? Ans. 3002.
3. What is the cube root of ,0001357? Ans. ,05138, &c.
4. What is the cube root of $\frac{1520}{5130}$? Ans. ,67, &c.
5. What is the cube root of $\frac{2}{3}$? Ans. ,873, &c.
6. The statute bushel contains 2150,4197, &c. cubic inches; what is the side of a cubic box which shall contain the same quantity? Ans. 12,907, &c. inches.
7. A cube of silver, whose side is 2 inches is worth 20 dollars; what will be the side of a cube of silver whose value shall be 8 times as much? Ans. 4 inches.

NOTE.—To solve such questions as the preceding, *cube the given side, and multiply it by the proportion between the given and required cube, and the cube root of the product will be the required side.*

8. There is a cubical vessel whose side is 4 feet; what will be the side of another cubical vessel that shall contain 4 times as much? Ans. 6,349, &c. feet.
9. A cooper, who has a cask 40 inches long, and 32 inches at the bung diameter, is required to make another cask of the same shape, but which shall hold just twice as much; what will be the length of the new cask, and its bung diameter?
Ans. 50,3, &c. inches length; and 40,3, &c. in. bung diam.
10. What is the cube root of 12,977875? Ans. 2,35.
11. What is the cube root of 36155,027576? Ans. 33,06, &c.
12. What is the cube root of ,001906624? Ans. ,124.
13. What is the cube root of $12\frac{19}{37}$? Ans. 2,3, &c.
14. What is the cube root of $31\frac{15}{343}$? Ans. 3,1.

15. What is the cube root of $7\frac{1}{8}$? Ans. 1,93, &c.

16. What is the cube root of $9\frac{1}{6}$? Ans. 2,092, &c.

17. If a block of marble be 47 inches long, 47 inches broad, and 47 inches deep, how many cubical inches does it contain? Ans. 103823.

18. A cellar was dug 12 feet long, 12 feet broad, and 12 feet deep; how many solid feet of earth were taken out of it? Ans. 1728.

19. How many cubes of 3 inches are contained in a cubical foot? Ans 64.

20. A certain stone of a cubical form contains 474552 solid inches; what is the superficial contents of one of its sides? Ans. 6084 inches.

21. What are the two mean proportionals between 6 and 162? Ans. 18 and 54.

NOTE.—To solve such questions as the last, *divide the greater extreme by the less, and the cube root of the quotient multiplied by the less extreme gives the less mean; multiply the same cube root by the less mean, and the product will be the greater mean proportional.*

22. What are the two mean proportionals between 4 and 108? Ans. 12 and 36.

23. What is the cube root of 389017? Ans. 73.

24. What is the cube root of 32461759? Ans. 319.

25. What is the cube root of 259694072? Ans. 638.

26. What is the cube root of 22069810125? Ans. 2805.

27. What is the cube root of ,001906624? Ans. ,124.

28. What is the cube root of 219365327791? Ans. 6031.

29. What is the cube root of 15926,972504? Ans. 25,16, &c.

30. What is the cube root of 673373097125? Ans. 8765.

RULE FOR EXTRACTING THE ROOTS OF POWERS IN GENERAL.

Prepare the given number by separating it into divisions of as many figures as the root required directs.

Find the first figure of the root by trial, and subtract its power from the given number.

To the remainder bring down the first figure of the next division, and call it the dividend.

Involve the root, which has been found, to the power next below the given one, and multiply it by the number expressing the given power, for a divisor.

Find how many times the divisor is contained in the dividend, and the quotient will be another figure of the root.

Involve the whole root to the given power, and subtract it from the given number as before.

Bring down the first figure of the next division, to the remainder, to form a new dividend; to which find a new divisor as before; and so on, till the whole root be extracted.

EXAMPLES FOR PRACTICE.

1. What is the fourth root of 19987173376? Ans. 376.
2. What is the fifth root of 307682821106715625? Ans. 3145.
3. What is the cube root of 53157376? Ans. 376.
4. What is the sixth root of 435728-81009267809889764416? Ans. 27534.

RATIOS, PROPORTIONS, PROGRESSIONS, AND RULES, WHICH DEPEND UPON THEM.

162. In Mathematics, the words *ratio* and *relation* have the same signification, both expressing the result of the comparison of two quantities.

163. If in the comparison of two quantities the object be to know how much the one surpasses or is surpassed by the other, the result of this comparison, which is the difference of the two quantities, is called their *Arithmetical Ratio*. Thus, if we compare 15 with 8 to know what is their difference, the number 7, which is the result of this comparison, is the arithmetical ratio of 15 to 8. To show that two quantities have been compared in this manner, we separate them by a point; thus 15 · 8 signifies that we have considered the arithmetical ratio of 15 to 8.

164. If in the comparison of two quantities the object be to know how many times one contains or is contained in

the other, the result of this comparison is called their *Geometrical Ratio.* For example, if we compare 12 with 3 to know how many times 12 contains 3, the number 4, which expresses the number of times, is the geometrical ratio of 12 to 3. To show that two quantities have been compared in this manner, we separate them by two points. This expression 12 : 3 signifies that we have considered the geometrical ratio of 12 to 3.

165. When two quantities have been compared, the first which is written or enunciated is called the *antecedent,* and the other, *the consequent.* Thus, in the ratio, 12 : 3, 12 is the antecedent and 3 the consequent, and both are called the *terms* of the ratio.

166. The arithmetical ratio of two quantities is found by subtracting the less from the greater.

167. The geometrical ratio of two quantities is found by dividing one by the other.

168. This ratio will be determined hereafter by dividing the antecedent by the consequent. Thus, the ratio of 12 to 3 is 4, and the ratio of 3 to 12 is $\frac{3}{12}$ or $\frac{1}{4}$.

169. An arithmetical ratio is not altered by adding the same quantity to both of its terms, or by subtracting the same quantity from them; because the difference, in which the ratio consists, remains the same.

170. A geometrical ratio is not altered by multiplying or dividing both of its terms by the same number; because the geometrical ratio consisting (168) in the quotient of the division of the antecedent by the consequent, is a fractional quantity which is not changed (88) by multiplying or dividing both its terms by the same number. Thus, the ratio 3 : 12 is the same as that of 6 : 24, which is obtained by multiplying the terms of the first by 2. It is also the same as that of 1 : 4, which is found by dividing them by 3.

171. This property is useful for simplifying ratios. For example, if we wish to examine the ratio of $6\frac{3}{4}$ to $10\frac{2}{3}$, we say, reducing the whole to a fraction, it is the same as that of $\frac{27}{4}$ to $\frac{32}{3}$; or, by reducing them to a common denominator, the same as that of $\frac{81}{12}$ to $\frac{128}{12}$; or by suppressing the denominator 12, which is the same as to multiply the terms of the ratio by 12, the ratio is the same as that of 81 to 128.

172. When four quantities are such, that the ratio between the two first is the same as the ratio between the two last, these four quantities form a *proportion*; and the proportion is either arithmetical or geometrical, according as the ratio in the proportion is arithmetical or geometrical.

The four quantities 7,9,12,14 form an arithmetical proportion; because the difference between the two first is the same as that between the two last. To show they form an arithmetical proportion, we write them thus, $7 \cdot 9 : 12 \cdot 14$; that is, we separate the terms of each ratio by a point, and the two ratios, by two points. The point which separates the terms of each ratio, signifies *is to*; and the two points, which separate the two ratios, signify *as*. Therefore, when the proportion, written above, is enunciated, we say 7 *is to* 9 *as* 12 *is to* 14.

The four quantities 3,15,4,20, form a geometrical proportion, because 3 is contained in 15 as often as 4 is contained in 20. To show they form a geometrical proportion, we write them thus, $3 : 15 :: 4 : 20$; that is, we separate the terms of each ratio by two points, and the two ratios, by four points. The two points signify *is to*, and the four points signify *as*. Hence, we say 3 *is to* 15, *as* 4 *is to* 20.

The better to distinguish between an arithmetical and geometrical proportion, in the enunciation let the word *as* precede the word *arithmetically* or *geometrically*, as the case may be. Thus, in an arithmetical proportion, 7 is to 9 as (arithmetically) 12 is to 14; and in a geometrical proportion, 3 is to 15 as (geometrically) 4 is to 20.

173. The first and last terms of a proportion are called the *extremes*; the second and third, or two middle terms, are called the *means*.

As there are two ratios, and consequently two antecedents and two consequents, we say, for the first ratio, *first antecedent, first consequent*; and for the second ratio, *second* antecedent, *second consequent*.

174. When the two mean terms of a proportion are equal, the proportion is called a *continued* proportion. $3 \cdot 7 : 7 \cdot 11$, form a continued arithmetical proportion, and are written in this manner $∺ 3 \cdot 7 \cdot 11$. The four points and bar which precede the proportion, show, that in the enunciation, the mean term, which in this case is 7, must be repeated.

The proportion 5 : 20 :: 20 : 80 is a continued geometrical proportion, and by abbreviation is written thus, ∺ 5 : 20 : 80. The use of the four points and bar is the same as in an arithmetical continued proportion.

175. It follows from what has been said concerning geometrical and arithmetical proportions. 1st. That if in an arithmetical proportion we add, to each of the antecedents or subtract from each, the ratio or difference, which subsists between the terms of the proportion, according as the antecedent is less or greater than its consequent, then each antecedent will become equal to its consequent; because we give to the smaller term of each ratio what is wanting to make it equal to the greater, or subtract from the greater that by which it surpasses the less. Thus in the proportion 3 · 7 : 8 · 12, add to the first and third terms, the number 4, which is the ratio or difference between the antecedents and consequents of the two terms, and we have 7 · 7 : 12 · 12. It is plain, this is a general property. 2nd. If in a geometrical proportion we multiply each of the two consequents by the ratio, they will become equal, each to its antecedent; for to multiply the consequent by the ratio, is to take the consequent as many times as it is contained in the antecedent. Thus, in the proportion 12 : 3 :: 20 : 5, if we multiply 3 and 5, each by 4, we shall have 12 : 12 :: 20 : 20. In like manner, in the proportion 15 : 9 :: 45 : 27, if we multiply 9 and 27, each by $\frac{15}{9}$ or $\frac{5}{3}$, which is the ratio, we shall have 15 : 15 :: 45 : 45.

PROPERTIES OF ARITHMETICAL PROPORTIONS.

176. The fundamental property of an arithmetical proportion is, that *the sum of the extremes is equal to the sum of the means*. For example, in the proportion 3 · 7 : 8 · 12, the sum of the extremes 3 and 12, is 15, and it is equal to the sum of the means 7 and 8. It is plain, this is a general property. For if the two first terms of the proportion are equal, and also the two last, as in this proportion,

7 · 7 : 12 · 12,

the sum of the extremes will evidently be equal to that of the means. But every arithmetical proportion can be reduced to this condition (175), by adding to each antecedent or subtracting from it, the difference between the terms of the proportion. This addition, which augments equally the sum of the means and the sum of the extremes, cannot change the equality of their two sums. Hence, if they become equal by this addition, they were equal before the addition was made. The reasoning is the same in case of subtraction.

177. Since, in a continued proportion the two mean terms are equal, it follows from what has been shown, that the sum of the extremes must be double the mean term; or, the mean term is half the sum of the extremes. Thus, to find the mean term between 7 and 15, we add 7 to 15, and taking half their sum, have 11 for the mean term. The proportion is then written in this manner, ∺ 7 · 11 · 15.

PROPERTIES OF GEOMETRICAL PROPORTIONS.

178. The fundamental property of a geometrical proportion is, that *the product of the extremes is equal to the product of the means*. For example, in this proportion 3 : 15 :: 7 : 35, the product of 35 by 3, which is 105, is equal to the product of 15 by 7. It is plain this property is common to every geometrical proportion. For if the antecedents are equal to their consequents as in this proportion 3 : 3 :: 7 : 7, the product of the extremes will evidently be equal to the product of the means. But every proportion can be reduced to this condition (175), by multiplying the two consequents by the ratio. It is true, this multiplication will make the product of the extremes a certain number of times greater than it would have been, if no multiplication had been made; or, if the ratio be a fraction, a certain number of times less; but the product of the means will, at the same time, be proportionably augmented or diminished. Then, since after multiplication, the product of the extremes is equal to the product of the means, these two products must have been equal before the multiplication. The product of the extremes and means may then be taken reciprocally for each other.

And *in a continued proportion, the product of the extremes is equal to the square of the mean term:* because, the two mean terms being equal, their product is the square of either. Then, to find a geometrical mean between two given numbers, we multiply the numbers together and extract the square root of their product. Thus, to find the geometrical mean between 4 and 9, we multiply 4 by 9, and from 36 their product, we extract the square root 6, which is the mean proportional sought.

179. It follows from the fundamental property of a geometrical proportion, that if three terms of a proportion are known, *the fourth may be found by multiplying the second by the third and dividing their product by the first.* For it is plain (174) the fourth term may be found by dividing the product of the extremes by the first term. But this product is the same as the product of the means. Then the fourth term may also be found by dividing the product of the means by the first term. Thus, if the fourth term of a proportion be required, the three first of which are 3 : 8 :: 12; we multiply 8 by 12, and divide the product 96 by 3; the quotient 32 is the fourth term sought. Hence 3,8,12,32, form a proportion; and the first ratio is $\frac{3}{8}$ and the second $\frac{12}{32}$, which (89) by dividing both terms by 4, is also $\frac{3}{8}$.

By similar reasoning, any other term of a proportion may be found, when three terms are already known. *When the term required is an extreme, the two mean terms are multiplied together and their product divided by the known extreme; on the contrary, if a mean term is required, the extremes are multiplied together and their product divided by the known mean term.*

180. This property of the equality between the product of the extremes and the product of the means, can belong only to four quantities in geometrical proportion. Hence if we have four quantities which do not form a geometrical proportion, on multiplying the consequents by the ratio between the two first terms, the first antecedent only will become equal to its consequent. For example, if we have the numbers 3,12,5,10; after multiplying the consequents 12 and 10 by $\frac{1}{4}$, the ratio between two first terms, we then have 3,3,5,$\frac{10}{4}$, in which it is plain, that the product of the extremes cannot be equal to the product of the means. These products then would not have been equal, if the consequents had been multiplied by the ratio $\frac{1}{4}$. It is plain, this reasoning may be applied to all other cases.

Then, *if four quantities are such, that the product of the extremes is equal to the product of the means, these four quantities form a proportion.* From this principle is deduced the following second property of proportions.

181. *If four quantities are in proportion, they will continue in proportion, although the extremes be put in place of the means, and the means in place of the extremes.*

182. The same property remains, that is, *the proportion still continues, although one extreme or one mean be made to occupy the place of the other.*

And in all cases it is easy to prove, that the product of the extremes is equal to the product of the means. Thus the proportion 3 : 8 :: 12 : 32. by a single change of its terms, furnishes the following proportions.

3 : 8 :: 12 : 32
3 : 12 :: 8 : 32
32 : 12 :: 8 : 3
32 : 8 :: 12 : 3
8 : 3 :: 32 : 12
8 : 32 :: 3 : 12
12 : 3 :: 32 : 8
12 : 32 :: 3 : 8

Any other proportion may be treated in the same manner.

183. In as much as the third term may be put in place of the second, and reciprocally, it is inferred *that the two antecedents may be multiplied or divided by the same number without changing the proportion; and also the two consequents.* For in making this change, the two antecedents of the given proportion will form the first ratio; and the two consequents, the second ratio. Then, to multiply the two antecedents of the first proportion, is the same thing as to multiply the two terms of a ratio, each by the same number, which does not (170) change that ratio. For example, if we have the proportion 3 : 7 :: 12 : 28; then, after dividing the two antecedents by 3, we say 1 : 7 :: 4 : 28; because, from the proportion 3 : 7 :: 12 : 28, we may infer (182) that 3 : 12 :: 7 : 28; and after dividing the terms of the first ratio by 3; 1 : 4 :: 7 : 28; which (182) may be changed to 1 : 7 :: 4 : 28.

184. *Every change made in a proportion, in such manner, that the sum of an antecedent and its consequent, or*

their difference, may be compared with the antecedent or consequent, (after the same manner in each ratio), will form a proportion.

For example, if we have the proportion

12 : 3 :: 32 : 8

we can deduce from it the following proportions.

12 *plus* 3 : 3 :: 32 *plus* 8 : 8
or 12 *minus* 3 : 3 :: 32 *minus* 8 : 8
or 12 *plus* 3 : 12 :: 32 *plus* 8 : 32
or 12 *minus* 3 : 12 :: 32 *minus* 8 : 32

For, if the consequent be compared with its antecedent, it is plain that the antecedent, augmented or diminished by the consequent, will contain the consequent once more, or once less, than before. And as this comparison is made in the same manner for the second ratio, which, from the nature of a proportion, is equal to the first, it necessarily follows, that the two new ratios will also be equal.

If the antecedent be compared with its consequent, the same reasoning is followed, by supposing in the proportion upon which the change is made, that the antecedent of each ratio is put in place of its consequent, and the consequent in place of its antecedent (181).

185. Since, when the third term of a proportion is put in place of the second, and reciprocally, there is no change in the proportion (182), it may be inferred, that the two antecedents contain each other as often as the two consequents. Then, *the sum of the two antecedents in every proportion contains the sum of the two consequents, or is contained in it, as many times as one of the antecedents contains its consequent, or is contained in it.* For example, in the proportion

12 : 3 :: 32 : 8

12 plus 32 : 3 plus 8 :: 32 : 8. And to prove this a general property, we may observe, that if the first antecedent contain the second, four times for instance; the sum of the two antecedents will contain the second, five times, and for the same reason, the sum of the consequents will contain the second consequent 5 times. Then the sum of the antecedents will contain that of the consequents as often as the quintuple of one of the antecedents contains the quintuple of its consequent; that is (170), as

often as one of the antecedents contains its consequent. In like manner, it may be proved, that the difference of the antecedents is to the difference of the consequents, as one antecedent is to its consequent.

186. It is plain from the proposition which has been demonstrated, that if two equal ratios, for example that
of . . . . 4 : 12
And that of . . . . . 7 : 21
11 : 33
are added together, the sums of the two antecedents and two consequents will have the same ratio. Then, *if many equal ratios are given, the sum of all the antecedents is to the sum of all the consequents, as one antecedent is to its consequent.* For example, if we have the equal ratios 4 : 12 :: 7 : 21 :: 2 : 6, we can say 4 *plus* 7 *plus* 2, are to 12 *plus* 21 *plus* 6, as 4 is to 12, or as 7 is to 21, &c. For, after having added the antecedents of the two first ratios, and their consequents, the new ratio, (which, according to what has been shown, will be the same as that in each of the two first,) will also be the same as the third. Hence, if they be added together, the results will have the same ratio; and so on.

187. *A compound ratio* is the ratio, which results from two or more ratios, whose antecedents and consequents are multiplied together. For example, if we have the two ratios 12 : 4 and 25 : 5, 300 will be the product of the antecedents 12 and 25; and 20 will be the product of the consequents 4 and 5. Then the ratio of 300 to 20 is what is called the compound ratio of 12 to 4 and of 25 to 5.

188. This ratio is the same as that which would have resulted, if each of the composing ratios had been taken separately, and the numbers composing them multiplied together. Thus the ratio of 12 to 4 is 3, that of 25 to 5 is 5; but, three times 5 make 15, which is the ratio of 300 to 20. That this is a general property may be seen if we observe, that (168) the ratio of two quantities is measured by a fraction, of which the antecedent is the numerator and the consequent the denominator. Then, the compound ratio must be a fraction whose numerator is the product of both the antecedents, and whose de-

nominator is the product of both the consequents; it is therefore, the product of two fractions, which express the composing ratios.

189. If two equal ratios are multiplied together, the compound ratio is called the *double ratio;* if three equal ratios are multiplied together, *the triple* ratio; and the *quadruple ratio*, when four equal ratios are multiplied together, and so on. For example, if we multiply the ratio of 2 to 3 by that of 4 to 6, which is equal to it, we shall form the compound ratio of 8 : 18, which is called the *double* ratio of the ratio of 2 to 3 or of 4 to 6.

190. *If two proportions are multiplied in order; that is, the first term of one by the first term of the other, the second by the second, and so on; the four resulting products will be in proportion.* For this multiplication is the multiplication of two equal ratios by two equal ratios (172); then the compound ratios, which result, must be equal; the products, therefore, must be in proportion (172).

191. Hence it is inferred, *that the squares, cubes, and in general, the like powers of four quantities in proportion, are also in proportion.* Because, to form these powers, nothing is required but the successive multiplication of the proportion by itself a certain number of times.

192. *The square roots, cube roots, and in general, the like roots of four quantities in proportion, are also in proportion.* Because, the ratio of the square roots of the two first terms, is the same as the square root of the ratio of these terms (142 and 167); the same is true of the ratio of the square roots of the two last terms. Then, as the two primitive ratios are supposed to be equal, their square roots will also be equal; hence, the ratio of the square roots of the two first terms will be equal to the ratio of the square roots of the two last terms. The same property belongs to the cube and higher roots.

USES OF THE PRECEDING PROPOSITIONS.

193. The propositions, which have been demonstrated, and which are called the *Rules of Proportions*, have continual application in all branches of Mathematics. But in this treatise we shall confine ourselves to such as pertain to Arithmetic, and commence with the proposition laid down in article (1.9), which is the foundation upon which most of the others depend.

SIMPLE RULE OF THREE DIRECT.

194. There is a distinction made of the rules of *Three* into several kinds; but they all have for their object to find one term in a proportion, of which three are already known. *The simple Rule of Three Direct* is called *simple*, because the enunciation of the questions, to which it applies, comprehends only four terms, three of which are known, and the fourth is to be found. It is called *direct*, because, of the four quantities considered, there are always two, which are not only relative quantities to two others, but which depend upon them in such manner, that one of these quantities contains the other or is contained in it, as often as the quantity relative to the first contains the quantity relative to the second, or is contained in it. The following is a more abridged method of expressing the same principle; *any quantity and its relative may always be, reciprocally, either antecedent or consequent in a proportion.* This is not a property of the rule of Three Inverse, as will be seen hereafter.

The method for finding the fourth term of a proportion, and, consequently, for performing the Simple Rule of Three Direct, as explained in article (179), is as follows; *Multiply the second and third terms together and divide their product by the first; the quotient will be the answer, or fourth term sought.* The following examples will explain the various uses of this rule.

EXAMPLE I.

In a certain time 40 labourers built 268 rods of wall; it is required to know how much 60 labourers can build in the same time?

It is plain, that the number of rods must be augmented in proportion to the number of labourers; wherefore, if the number of labourers be doubled, tripled, &c. the number of rods must be doubled, tripled, &c. Hence, the number of rods sought must contain the given number 268 rods, as often as the number 60, which is relative to the first, contains the number 40, relative to the second. Therefore the fourth term of a proportion must br sought, which commences with these three;

40 : 60 :: 268 R :

Or, (by dividing the two first terms, each by 20), which is permitted (170,) with these three;

2 : 3 :: 268 R. :

Then, according to the direction in article (179), we multiply 268 R. by 3, and divide the product 804 by 2, which gives for quotient 402 R.; and consequently, 402 rods for the wall built by 60 labourers.

EXAMPLE II.

A ship with the same wind sailed 275 leagues in 3 days; it is required to know in what time she will sail 2000 leagues, all circumstances being equal?

The length of time will evidently be in proportion to the number of leagues; then the number of days sought must contain 3 days, as often as 2000 leagues contain 275 leagues. Hence the fourth term of a proportion must be sought, which commences with the three following . . . . . . 275 : 2000 :: 3 :

Multiplying 2000 by 3 and dividing the product 6000 by 275, we have $21\frac{9}{11}$ days for the answer, or fourth term.

EXAMPLE III.

If 52F. 4ft. 5in. of work cost £168 9s. 4d., what ought 77F. 1ft. 8in. to cost?

The price of 77F. 1ft. 8in. must contain £168 9s. 4d., the price of 52F. 4ft. 5in., as many times as 77F. 1ft. 8in. contain 52F. 4ft. 5in. We must then seek the fourth term of a proportion, which commences with these three;

52F. 4ft. 5in. : 77F. 1ft. 8in. :: £168 9s. 4d. :

That is, we must multiply £168 9s. 4d. by 77F. 1ft. 8in. and divide the product by 52F. 4ft. 5in., which is accomplished by following the direction in articles (122 and 128). But the simplest method of operation is to reduce the two first terms to units of the lowest kind contained in them, that is, to inches; the question then is to find the fourth term of a proportion, which commences with these three;

3797 : 5564 :: £168 9s. 4d. :

Then multiplying £168 9s. 4d. by 5564 we have £937348 10s. 8d.; dividing this sum by 3797, the quotient £246 17s. 3d. $\frac{2789}{3797}$ will be the price to be paid for 77F. 1ft. 8in.

If fractions are in the given sum, after having reduced the two terms, which are of the same kind, to their lowest denomination, as in the last example, the ratio of these two terms may be simplified in the manner explained in article (171).

EXAMPLES FOR PRACTICE.

1. If 2 yards of muslin cost 4 shillings, what will 6 yards cost? Ans. 12.

2. If 10 shillings will pay for 20 pounds of beef, how many pounds will 5s. pay for? Ans. 10 lbs.

3. If 1 pound of sugar cost 9d. what will 2cwt. 2qrs. 10lb. cost? Ans. £10 17s. 6d.

4. If 12 bushels of wheat be worth $16, how much are 48 bushels worth? Ans. $64.

5. What will 128 pounds of pork come to at 8 cents a pound? Ans. $10,24.

6. What is the value of a pipe of wine at 10d. 2qrs. per pint? Ans. £52 17s.

7. What is the value of 1cwt. 2qrs. of coffee at 5d. 2qrs. per ounce? Ans. £61 12s.

8. How much will 75 pounds of almonds come to at 37½ cents per pound? Ans. $28,12½.

9. If 1 dozen of penknives cost $2,50, what will 4 dozen come to? Ans. $10.

10. What is the tax upon £745 14s. 8d. at 3s. 6d. in the pound? Ans. £130 10s. 0¾d. $\frac{48}{240}$.

11. What is the value of 2qrs. 1na. of velvet at 19s. 8d. 2qrs. per ell English? Ans, 8s. 1d. 1qr. $\frac{14}{20}$.

12. At $5,50 per thousand, what will 37 thousand of boards come to? Ans. 203,50.

13. If 3 and 4 make 9, what will 6 and 8 make? Ans, 18.

14. How many quarters of corn may be purchased for $140, at 4s. a bushel? Ans. 26 qrs. 2 bu.

15. If 1cwt. of iron be worth £1 8s, what is 33cwt. 1qr. 22lb. worth? Ans. £46 16s. 6d.

16. B owes £2119 17s. 6d. and he is worth but £1324 18s. 5d. 1qr.; if he deliver this to his creditors, what do they receive on the pound? Ans. 12s. 6d.

17. A merchant failing in trade owes in all $29475 and delivers up his whole property, worth $21894,3; how much per cent. does he pay, and what is B's loss to whom he owed $325? Ans. He pays $74,28 per cent.
And B. loses $83,59cts.

18. A silver smith bought 73lb. 5oz. 15dwt. of silver, for which he paid 5s. 9d. per ounce; what was the amount? Ans. £253 9s. 4d. 3qrs.

19. A French crown is 8s. 3d. Pennsylvania currency, how many pounds of that currency are 100 French crowns? Ans. £41 5s.

20. If $100 gain $6 interest in one year, how much will $480 gain in the same time? Ans. 28,80cts.

21. If a man spend 46 cents a day, what will it amount to in a year? Ans. 167,90cts.

22. When a man's yearly income is $949, what is it per day? Ans. 2,60cts.

23. What will a hogshead of sugar come to weighing 7cwt. 3qrs. at $10,62cts. 5m. per hundred weight?
Ans. $82,34$\frac{1}{2}$cts.

24. A captain of a ship is provided with 18000lb. of bread for 150 seamen, of which each man eats 4lb. per week; how long will it last them? Ans. 30 weeks.

25. Bought 4 pieces of cloth, 2 of which contained $27\frac{1}{2}$ yards each, and the other two $25\frac{3}{4}$ yards each, at $62\frac{1}{2}$ cents per yard; what was the cost? Ans. $66,56$\frac{1}{4}$cts.

26. If a person having a salary of $1333 a year, spend $2,14cts. a day, how much will he save in a year?
Ans. $551,90cts.

27. How long will 2295lb. of beef last 45 seamen if they receive 1lb. each, three times a week?
Ans. 17 weeks.

28, If a staff 4 feet long cast a shade on level ground 7 feet, what is the height of a steeple whose shade is 198 feet? Ans. $113\frac{1}{7}$ feet.

29. If a man's income be $890,50cts. a year; how much may he spend each day to save $120,56cts. every year? Ans. $2,11[illegible]

30. Bought 40 tubs of butter weighing 36cwt. 2qrs. 14lb. net, for $472,2cts. paid cooperage 12cts. per tub; salt and labour $4,83cts. 8mills; storage $6,48 cts. What was the cost of the butter per pound?
Ans. 11cts. 9m.

31. What will 12 gallons of oil come to, if 6 gallons cost 18s. Ans. 36s.

32. What will 4hhd. of rum come to, containing as follows, the first 101½; the second 96¾, the third 89¼, and the fourth 111¾ gallons; if 6½ gallons cost $6,89 cts.? Ans. 423,20cts. 5m.

33. If 1000 feet of boards are worth $11,12cts.; what will be the price of 17221 feet?
Ans. $191,49cts. 7 &c. m.

34. If 6½ dozen of pigeons are worth 65cts; what will be the price of 29¾ dozen? Ans. $2,97ts. 5m.

35. If 65 bushels and 1 peck of corn were consumed in a family of 6 persons; what quantity would be sufficient to support a family of 22 persons for the same time?
Ans. 239¼ bushels.

36. How many times would a wheel 16ft. 3in. in circumference turn round in passing the globe upon the equator? Ans. 7015774 $\frac{79}{195}$.

37. If the earth make one revolution in a solar day, how long is it in passing one degree? Ans. 4 minutes.

38. If the sun is 4 minutes in passing one degree, what is the difference in time of its coming to the meridian at two places 20 degrees apart?
Ans. 1 hour 20 minutes.

39. What is the insurance upon $3472 at 3½ per cent?
Ans. 121,52cts.

40. If half an acre of land is worth $59,20cts., what is the price of 1¼ acres? Ans. $148.

41. What is the value of $642 against an estate which can pay only 69 cents on a dollar? Ans. $442,98cts.

42. There is a cistern having 4 cocks; the 1st empties it in 15 minutes; the 2d in 30 minutes; the 3d in 45 minutes; and 4th in 60 minutes; in what time would it be emptied if they were all running together?

15 : 1 :: 120 : 8
30 : 1 :: 120 : 4
45 : 1 :: 120 : $2\frac{2}{3}$
60 : 1 :: 120 : 2

$16\frac{2}{3}$

cis. min. cis.
$16\frac{2}{3}$: 120 :: 1 : 7 min. 12 sec. Ans.

43. If £21 12s be paid for building 60 rods of stone wall, how much must be paid for the building of 96 rods?
Ans. £34 $11\frac{1}{5}$d.

44. What must be given for $19\frac{1}{2}$ yards of broadcloth, if 3 yards of the like kind cost £19 11s. 6d.?
Ans. £127 4s. 9d.

45. If $\frac{3}{16}$ of a ship cost £273 2s. 6d., what is $\frac{5}{32}$ of her worth? Ans. £227 12s. 1d.

46. How much land at $2,50cts. per acre must be given in exchange for 360 acres, at $3,75cts. per acre?
Ans. 540 acres.

47. A man having a piece of land to plant, hired two men and a boy to plant it; one of the men could plant it in 12 days, the other in 15 days, and the boy in 27 days; in how long time might they plant it if they all worked together? Ans. 5,346 days.

48. A merchant sold a quantity of goods far $243,00, by which he gained $\frac{1}{8}$ of the first cost What was the first cost, and what did he gain? Ans. if he gained $\frac{1}{8}$ of the first cost, $243,00 are $\frac{9}{8}$ of the first cost; hence the first cost was $216, and the gain $27.

49. A merchant sold a quantity of goods for $187,00 by which he lost $\frac{1}{4}$ of the first cost. What was the cost of the goods, and what was the loss? Ans. If $\frac{1}{4}$ of the cost was lost, $187,00 are $\frac{3}{4}$ of the cost; hence the cost was 249,$33\frac{1}{3}$, and the loss $62,$33\frac{1}{3}$.

50. A merchant sold a bale of cloth for $347,00, by which he gained $\frac{3}{10}$ of what it cost him. How much did it cost him, and how much did he gain?
Ans. The cost was $266,92$\frac{4}{13}$, and gain $80,07$\frac{9}{13}$.

51. A merchant sold a quantity of goods for $948,00, by which he lost $\frac{4}{11}$ of the cost. How much did he lose?
Ans. $344,72$\frac{8}{11}$.

52. A merchant sold a quantity of goods for $273,00, by which he gained 10 per cent. on the first cost. How

much did they cost? Ans. 10 per cent. is 10 dollars on 100 dollars, or $\frac{0}{100}$; and 10 per cent. of the first cost is $\frac{10}{100}$ of the first cost. Therefore \$273,00 is $\frac{110}{100}$ of the first cost, which was \$248,18$\frac{2}{11}$.

53. A merchant sold a quantity of goods for \$135,00 by which he gained 13 per cent. How much did the goods cost, and how much did he gain?

Ans. Cost \$119,46$\frac{102}{113}$. Gained 15,53$\frac{10}{103}$.

54. A merchant sold a quantity of goods for \$3,875, by which he gained 65 per cent. How many dollars did he gain? Ans. \$1,526$\frac{17}{33}$.

55. A merchant sold a quantity of goods for \$983,00, by which he lost 12 per cent. How much did the goods cost, and how much did he lose? Ans. If he lost 12 per cent. or $\frac{12}{100}$, he must have sold for $\frac{88}{100}$ of the first cost which was \$1117,04$\frac{6}{11}$. The loss was \$134,4$\frac{6}{11}$.

56. A merchant sold a quantity of goods for \$43,00 more than they cost him, and by so doing gained 20 per cent. How much did the goods cost him? Ans. \$215.

57. A has tea which he sells B for 10d. per lb. more than it cost him, and in return B sells A cambrick which cost him 10s. per yard, for 12s. 6d. per yard. The gain on each was in the same proportion. What did A's tea cost him per pound? Ans. B. gains 2s. 6d. on a yard which is $\frac{1}{4}$ of the first cost; 10d. therefore must be $\frac{1}{4}$ of the first cost on the tea. The tea then cost 3s. 4d. per lb.

58. C has brandy which he sells to D for 20 cents per gallon more than it cost him; and D sells C molasses which cost 23 cents per gallon, for 32 cents per gallon, by which D gains in the same proportion as C. How much did C's brandy cost him per gallon? D gains 9 cents on a gallon, which is $\frac{9}{23}$ of the cost; hence 20cts. is $\frac{9}{23}$ of the brandy.

Ans. \$0,51$\frac{1}{9}$ per gallon.

59. A man having put a sum of money at interest at 6 per cent., at the end of 1 year received \$13 for interest. What was the principal? Ans. since 6 per cent. is $\frac{6}{100}$ of the whole, \$13 must be $\frac{6}{100}$ of the principal, which is \$216,66$\frac{2}{3}$.

60. What sum of money put at interest for 1 year will gain \$57, at 6 per cent.? Ans. \$950.

61. What sum of money put at interest at 6 per cent. will gain $53 in 2 years? Ans. $441,66⅔.

Note.—6 per cent. for 1 year, will be 12 per cent. for 2 years, 3 per cent. for 6 months, 1 per cent. for 2 months, &c.

62. What sum of money put at interest at 7 per cent. will gain $97 in 1 year and 6 months?

Ans. $1077,77 7/9.

63. What sum of money put at interest at 8 per cent. will amount to $137 in 2 years and 6 months?

Ans. $114,16⅔.

SIMPLE RULE OF THREE INVERSE.

195. The simple *Rule of Three Inverse* differs from the Rule of Three Direct in this, that of the four quantities which are considered in the enunciation of the question which requires this operation, the two principal terms contain each other in a contrary order to that of the two other terms which are relative to them. Wherefore, after an examination of the question, these four quantities are disposed in such a manner as to form a proportion, by having one of the principal terms and its relative placed as the extremes; the other principal term and its relative forming the means. This arrangement does not cause any difference in the method of operation; the object of it being always to find the fourth term of a proportion, or at least, being always reducible to that operation.

Some Arithmeticians have prescribed, for the case under consideration, a rule subject to the enunciation of the question; we do not follow their example. It is the nature of the question, and not its enunciation (which is often erroneous), that must direct us in the solution.

EXAMPLE I.

30 men have performed a piece of work in 25 days; how many men can perform the same work in 10 days?

In this second case, it is plain, that as many more men

19. If a field feed 6 cows 91 days, how long will it feed 21 cows? Ans. 26 days.

20. There is a cistern having 1 pipe which will empty it in 10 hours; how many pipes of the same capacity will empty it in 24 minutes? Ans. 25 pipes.

21. Lent a friend $292 for 6 months; sometime afterwards he lent me $806: how long may I keep it to balance the favour? Ans. 2 mo. 5 days.

22. If $\frac{5}{8}$ of a gallon of brandy cost 95 cents, what will $\frac{1}{8}$ cost? Ans. 0,19cts.

23. If $\frac{4}{5}$ of a hogshead of molasses cost $16,00, what will the whole hogshead cost? Ans. $20,00.

24. If $\frac{3}{8}$ of a yard of broadcloth cost $6,00, what will $\frac{1}{8}$ cost? Ans. $2,00.

25. A smith bought $\frac{5}{18}$ of a ton of Russia iron for $25,35; what would be the price of a ton at that rate? Ans. $91,26.

COMPOUND RULE OF THREE.

196. In the Rules of Three, which have already been explained, the quantity sought, and quantity of the same kind, expressed in the enunciation, have to each other a simple ratio, and determined by that of two other quantities, which, in like manner, are expressed in the enunciation of the question. But in the Compound Rule of Three, the ratio of the quantity sought to the quantity of the same kind, expressed in the enunciation, is not determined by the simple ratio of two other quantities only, but by many simple ratios, which are compounded (187) after the examination of the question. When the ratios have been compounded, the rule is reduced to a simple Rule of Three. The following examples will explain what has been said on this subject.

EXAMPLE I.

Thirty men have built 132 rods of wall in 18 days; how much will 54 men build in 28 days?

In this question, it is plain, that the answer does not depend upon the number of men only, but also upon the number of days.

To have regard to both, we must consider, that 30 men labouring 18 days will not perform more work than 18 times 30, that is, 540 men, who labour during one day. In like manner, 54 men labouring 28 days, will not accomplish more than 28 times 54, or 1512 men, who labour one day. The question is then changed to this: 540 men have built 132 rods of wall, how much will 1512 men build in the same time? That is, we must seek the fourth term of a proportion, which commences with these three;

hours. hours. rods.
540 : 1512 :: 132 :

Multiplying 1512 by 132, and dividing the product by 540, we find for answer to the question, 169$\frac{3}{5}$ rods.

EXAMPLE II.

A man walking 7 hours in a day, was 30 days traveling 230 leagues; if he walk 10 hours in a day, how many days will it take him to go 600 leagues, traveling always with the same speed. If he walk the same number of hours a day in each case, it is plain the increase of time will be in proportion to the increase of distance; but as in the second case, he travels each day a greater number of hours, less time will be required in performing the journey. Hence the operation to be performed requires two operations, one in the Rule of Three Direct, the other in the Rule of Three Inverse. The question, however, may be reduced to a question in the simple Rule of Three, by considering, that to travel 30 days, 7 hours in a day, is to travel 30 times 7, or 210 hours; hence the question is changed to this; if 230 leagues have been travelled in 210 hours, how long will it take to travel 600 leagues? When the number of hours, which is the answer to this question, has been found, the number of days demanded will be found by dividing that sum by 10, because the man travelled ten hours in a day. Hence the fourth term must be sought of a proportion, whose three first terms are the following;

leagues. leagues. hours.
30 : 600 :: 210 :

We find, that the fourth term required is 547$\frac{19}{23}$ hours; which, divided by 10, the number of hours the man travelled in a day, give 54$\frac{18}{23}$ days for the answer sought.

EXAMPLES FOR PRACTICE.

1. If 6 men in 8 days eat 10lb. of bread, how much will 12 men eat in 24 days? Ans. 60lbs.

2. If £100 in 12 months gain £5, how much will £400 gain in 3 months? Ans. £5.

3. If 4 men in 12 days mow 48 acres, how many acres can 8 men mow in 16 days? Ans. 128A.

4. If a family of 9 persons spend \$450 in 5 months, how much would be sufficient to maintain them 8 months, if 5 more were added to the family? Ans. \$1120.

5. If 10 bushels of oats will be sufficient for 18 horses 20 days, how many bushels will serve 60 horses 36 days, at the same rate? Ans. 60 bushels.

6. If 30 men perform a piece of work in 20 days, how many men will execute another piece, 4 times as large, in a fifth part of the time? Ans. 600 men.

7. What principal will gain \$315 in 7 years, at 6 per cent. per annum? Ans. \$750.

8. A wall, that is to be built to the height of 27 feet, has been raised 9 feet in 6 days, by 12 men; how many men will it take to complete the wall in 4 days? Ans. 36 men.

9. If the carriage of 8cwt. 128 miles cost \$12,80, what must be paid for the carriage of 4cwt. 32 miles? Ans. \$1,60cts.

10. If 6 men build a wall 20 feet long, 6 feet high, and 4 feet thick, in 16 days; in what time will 24 men build one 200 feet long, 8 feet high and 6 feet thick? Ans. 80 days.

$$\left.\begin{array}{r}24\text{ men}\\20\text{ feet long}\\6\text{ feet high}\\4\text{ feet thick}\end{array}\right\} : 16 :: \left\{\begin{array}{c}6\text{ men}\\200\text{ feet long}\\8\text{ feet high}\\6\text{ feet thick}\end{array}\right\} : 80$$

11. An usurer put out \$75 at interest, and at the end of 8 months received for principal and interest \$79; at what rate per cent. did he receive interest? Ans. 8 per cent.

12. If the freight of 12cwt. 2qrs. 6lb. 275 miles, cost \$27,78; how far may 60cwt. 3qrs. be carried for \$234,78? Ans. 480 miles.

13. If 4 men spend $60, going 300 miles; what will be sufficient for the expense of 20 men, and one boy 700 miles, allowing the boy one half a man's expenses?

Ans. $717,50.

14. If 50 men build a bridge in 144 days, what number of men will it take to build a like bridge in 270 days?

Ans. 10 men.

15. A man lent a friend $600, for 6 months, for which he received $9 interest; what sum will gain the same interest in 2 months? Ans. $1800.

RULE OF FELLOWSHIP.

197. The Rule of Fellowship is so called, because it is used for dividing among many associates the profit or loss of their partnership. Its object is to divide a given number into parts, which have known relations to each other. The rule used for this purpose is founded upon the principle established in article (186), that *if many equal ratios are given, the sum of all the antecedents is to the sum of all the consequents, as one antecedent is to its consequent.*

From this principle we deduce the following example.

EXAMPLE I.

It is required to divide 120 into three parts, which have to each other the same ratios as the numbers 4, 3, 2; the enunciation furnishes these two proportions.

4 : 3 :: as the first part, is to the second.
4 : 2 :: as the first part, is to the third.
Or (182) these other two proportions.
4 is to the first part :: 3 is to the second.
4 is to the first part :: 2 is to the third.

But it has been shown (186), that the sum of the antecedents of many equal ratios, is to the sum of the consequents, as one antecedent is to its consequent. Hence it is inferred, that 9, the sum of the three parts, which are proportional to those sought, is to 120, their sum, as any one of these three proportional parts, is to the part of 120, which answers to it.

The rule for performing this operation is as follows;

1st. *Add the given proportional parts into one sum.*

2d. *Make as many proportions in the Rule of Three as there are parts to be found.*

3d. *Let each proportion have for its first term, the sum of the given proportional parts, for its second term, the number proposed to be divided, and for the third term, one of the given proportional parts.* Thus in the question under consideration, the three following proportions must be made in the Rule of Three.

9 : 120 :: 4 :
9 : 120 :: 3 :
9 : 120 :: 2 :

Of which we find (179), that the fourth terms are $53\frac{1}{3}$, 40, and $26\frac{2}{3}$; which have to each other the ratios required, and together compose the number 120.

We may remark that it is not absolutely necessary to make as many applications of the Rules of Three as there are parts to be found; for the last may be dispensed with, by subtracting from the given number, the sum of the other parts when found.

EXAMPLE II.

Three persons are to share a prize vessel. The first has been at $20,000 cost in her capture; the second at $60,000; and the third at $120,000. The prize is valued at $800,000; how much ought each man to receive?

It is plain, the question requires $800,000 to be divided into parts which have to each other the same ratios as 20,000, 60,000, 120,000; or (170) as 2, 6, 12; because each person ought to receive in proportion to his expenditure. It is necessary, then, to add the three proportional parts 2, 6, 12, and make the three following proportions, or two only.

20 : 800,000 :: $ 2 : to the first part.
20 : 800,000 :: $ 6 : to the second part.
20 : 800,000 :: $12 : to the third part.

These three parts will be $80,000, $240,000, and $480,000.

More complicated questions may be solved on the same principles, as in the following example.

Three persons entered into partnership; the first put in $3000 for 6 months; the second, $4000 for 5 months;

the third, $8000 for 9 months. They gained $12050; what ought each person to receive?

Although the stocks were in trade unequal times, they are reduced to the same time, in the following manner. The stock $3000 ought to gain as much in 6 months, as 6 times $3000, or $18000 in one month. The stock $4000 ought to gain in 5 months, as much as 5 times $4000, or $20,000, in one month. And the stock $8000 ought to produce in 9 months, as much as 9 times $8000, or $72000, in one month. Hence the question is reduced to this; three partners put into trade $18000, $20,000, and $72000, and gain $12050; what ought each one to receive?

Proceeding as in the last example, we find $\$1971,81\frac{9}{11}$, $\$2190,90\frac{10}{11}$, and $\$7887,27\frac{3}{11}$, for their several shares.

REMARK ON THE PRECEDING RULE.

198. It may be useful here to examine a case, that might embarrass the learner. If it be required to divide the number 650 into three parts, of which the first shall be to the second :: 5 : 4, and the first to the third :: 7 : 3, we cannot apply the preceding rule without a preparation, which consists in making equal, in each given ratio, the parts which are proportional to one of the three parts sought. For example, that of the first; this is easily done by multiplying both terms of each ratio by the first term of the other ratio. Thus the two ratios 5 : 4 and 7 : 3, will have the same first term by multiplying both terms of the first by 7, and both terms of the second by 5; which does not change the value (170,) and gives the ratios 35 : 28 and 35 : 15. Hence the question becomes to divide the number 650 into three parts, which shall be to each other as the numbers 35, 28, and 15; which is easily done by means of the preceding rule.

If it be required to divide a number into four parts, of which the first is to the second : : 5 : 4, the first to the third : : 9 : 5, and the first to the fourth : : 7 : 3; these ratios may be reduced to have the same first term, by multiplying both terms of each by the products of the first terms of the other two. Thus, in this example we change the three given ratios into these other three, 315 : 252, 315 : 175, 315 : 135; hence the question becomes to divide the given number into four parts, which shall be to each other as the numbers 315, 252, 175, and 135.

EXAMPLES FOR PRACTICE.

1. A and B buy goods, which cost £120; of which A pays £80 and B £40, and they gain £32: what part of the gain ought each to receive?

Ans. A £21 6s. 8d.: and B £10 13s. 4d.

2. Three merchants trading together, gained $800, A's stock was $1200, B's $4800, and C's $2000; what was each man's share of the gain?

$120 A's share.
480 B's share. Ans.
200 C's share.

3. Three merchants trading together lost goods to the amount of 1920doll. A's stock in trade was 2880doll., B's 11520doll., C's 4800doll.; what share of the loss ought each man to sustain?

Ans. Doll.288 A's share.
1152 B's share.
480 C's share.

4. Three men are to share a legacy of $1500, of which B is to have $\frac{1}{2}$, C $\frac{1}{4}$, and D the remainder; but C relinquishes his share to B and D, to be divided according to their share in the whole. What part of the legacy will B and D resqectively receive?

Ans. B's part is $1000, and D's $500.

5. Divide the number 360 into 4 parts, that shall be to each other as the numbers 3, 4, 5, and 6.

Ans. 60
80
100
120

6. A and B enter into partnership; A put in £45, and took $\frac{3}{5}$ of the gain; what did B put in? Ans. £30.

7. A man dying left an estate of $12000, $\frac{1}{2}$ of which is to be given to his wife, $\frac{1}{3}$ to his eldest son, $\frac{1}{4}$ to his second son, and $\frac{1}{5}$ to his daughter. What share will each receive of the estate, divided in these proportions?

The numerators of the fractions, when reduced to a common denominator, are 30, 20, 15, and 12, which show the proportions. Their sum 77 is the first term of the several proportions to be made in the Rule of Three. Hence, Ans.

The eldest son's share is $3116,88$\frac{24}{77}$.
The second son's - 2337,66$\frac{18}{77}$.
The daughter's - - 1870,12$\frac{76}{77}$.
The wife's - - 4675,3$\frac{36}{77}$.

8. Two men A and B, traded in company. A put 350doll. for 8 months, and B 640doll. for 5 months; they gained 250doll. What was the share of each?

Ans { Doll.116,66$\frac{2}{3}$ A's share.
" 133,33$\frac{1}{3}$ B's share.

9. B and D joined stocks of $500 each; they traded 2 years, and B took out one fifth of his stock, and then continued their trade 3 years longer, and gained $627; what is the share of each?

Ans. { B's share $293,48$\frac{44}{47}$.
D's 333,51$\frac{3}{47}$.

10. A family of 10 persons took a house for $\frac{1}{2}$ a year, for which they were to pay $500. At the end of 14 weeks they took in 14 new lodgers, and after 3 weeks 4 more; and for every 3 weeks during the remainder of the term they took in 4 more lodgers. What part of the rent for the whole time must one of each class pay?

Ans. { One of the first class must pay $39, 09
" 2d " 12,167
" 3d " 8,046
" 4th " 4,841
" 5th " 2,219

11. Three graziers hired a piece of land for £60 10s. A put in 5 sheep for 4$\frac{1}{2}$ months, B put in 8 for 5 months, and C put in 9 for 6$\frac{1}{2}$ months; how much must each pay of the rent? Ans. A £11 7s. 5d. 3$\frac{125}{239}$; B £20, 5s. 5d. $\frac{11}{239}$, and C £28 17s. 1d. 3$\frac{103}{239}$.

12. Two men, K and L, were joint owners of a mill, in the building of which K laid out $150, and L $270. At the end of 7 months, K sold his share to L, and at the end of the year L sold the mill. The year's profit of the mill was $260; on settlement, what was each man's share?

K's share was $ 54,16$\frac{2}{3}$ } Ans.
L's " " 205,83$\frac{1}{3}$ }

OTHER RULES DEPENDING UPON PROPORTIONS.

199. The following rules are not only useful in themselves, but they also serve to explain the various applications of proportions.

SINGLE POSITION.

200. The first which we shall explain is called *Single Position.* It is used to solve questions which belong to the rules of Fellowship, from which it differs in this; that instead of taking parts proportional to those given in the enunciation of the question, it arbitrarily assumes one part to which the others are referred in conformity to the question. By this method calculation is rendered more easy.

EXAMPLE I.

Divide £640 between three persons, of which the second shall have four times as much as the first, and the third twice and $\frac{1}{3}$ as much as both the others. To represent the first part, we arbitrarily take the number 3, of which the $\frac{1}{3}$ part may be conveniently taken. The first part being 3, the second will be 12, and the third 35. The question then is to divide 640 into three parts, which shall be to each other as the three numbers 3, 12 and 35. This operation is performed by the rule in article (197).

The rule of Single Position is also used for solving such questions as are in some measure the reverse of those, which belong to the rule of Fellowship; for it teaches a method of descending from the sum of certain parts of a given number, to that number; as in the following example.

EXAMPLE II.

It is required to find a number, of which the $\frac{1}{3}$, $\frac{1}{5}$, and $\frac{3}{7}$, shall make 808. We take a number of which the $\frac{1}{3}$, $\frac{1}{5}$, and $\frac{3}{7}$, may be conveniently found. (This is easily done by multiplying the three denominators together.) This number is 105, of which we take the $\frac{1}{3}$, which is

35, the $\frac{1}{5}$, which is 21, and the $\frac{3}{7}$, which is 45. On adding these three numbers, we find their sum is 101, which is composed of the parts of 105, in the same manner as 808 is composed of the parts of the number in question; then this number must have the same ratio to 808 as 105 has to 101. It will therefore be the fourth term of a proportion, which commences with these three;

101 : 105 :: 808 :

The fourth term sought is 840, of which 808 contains the $\frac{1}{3}$, $\frac{1}{5}$, and $\frac{3}{7}$.

DOUBLE POSITION.

201. The second rule, of which we shall speak, is called *Double Position.* It is used for solving questions, in which it is not required to divide the given number, but only a part of it, into parts proportional to given numbers. The following example will explain this rule and its application.

EXAMPLE III.

It is required to distribute \$6954 among three persons, in such manner, that the second shall have as much as the first, and \$54 more; and the third as much as both the others, and \$78 more. If \$54 and \$78 were not in the question, it is plain, nothing would be required but to divide the given number into parts proportional to the numbers 1, 1 and 2; but as \$54 must be deducted for the second person, and \$54 plus \$78 for the third, only a part of the given number can be divided into parts proportional to 1, 1 and 2. For finding this part, which is easily done in the present example, but in others may be accomplished with more difficulty, the method here given is to be followed.

Suppose for the first part, any convenient number, \$1 for example; the second part will be \$1, plus \$54; that is \$55; and the third will be \$1 plus \$55, plus \$78; that is \$134: the sum of these parts is \$190. If it were only required to divide the given number into parts proportional to 1, 1 and 2; the first part being always supposed to be $1, the second would be $1, the third $2, and their sum $4; the difference of which with $190, is $186, the sum that must be deducted from the given sum $6954, which thus becomes reduced to $6768. It remains then to divide $6768 into parts proportional to 1, 1 and 2, according to the preceding rules; and having found that the first part is $1692, we conclude that the two other parts required are $1746 and $3516. The sum of these three parts is $6954.

EXAMPLES FOR PRACTICE UNDER THE TWO PRECEDING ARTICLES.

1. A person after spending $\frac{1}{3}$ and $\frac{1}{4}$ of his money had £60 left; how much had he at first? Ans, £144.

2. A schoolmaster being asked how many scholars he had, answered, "if to double the number I add $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ of them, I shall have 333;" what number of scholars had he? Ans. 108.

3. A certain sum of money is to be divided among 4 persons, in such a manner, that the first shall have $\frac{1}{3}$ of it, the second $\frac{1}{4}$, the third $\frac{1}{6}$, and the fourth the remainder, which is $30; what is the sum? Ans. $112.

4. What sum at 6 per cent. per annum, will amount to £360 in 12 years? Ans. £500.

5. What is the age of a person, who says, that "if $\frac{3}{5}$ of the years I have lived be multiplied by 7, and $\frac{2}{3}$ of them be added to the product, the sum of them will be 292?" Ans. 60 years.

6. A said to B, "my horse and saddle are together worth $150, but my horse is worth 9 times as much as the saddle." What was the value of each?

Ans, the horse is worth $\frac{9}{10}$ of $150, or $135;
and the saddle is worth $\frac{1}{10}$ of it, or $15.

7. A man dying left an estate of $13000, to four sons, in such a manner, that the third was to have once and one half as much as the fourth, the second was to have as much as the third and fourth, and the first was to have as much as the other three. What was the share of each? If the fourth has 2 parts, the third will have 3 parts, the second 5 parts, and the first 10 parts; the

sum of these parts is 20; hence the fourth will have $\frac{2}{20}$, the third $\frac{3}{20}$, the second $\frac{5}{20}$, and the first $\frac{10}{20}$ of $13000.

Ans. The share of the first is Doll. 6500.
" the 2d " 3250.
" the 3d " 1950.
" the 4th " 1300.

8. The stock of a manufacturing company is divided into 32 shares, and owned equally by 8 persons, A, B, C, D, &c. A sells 3 of his shares to a ninth person, who then becomes a member of the company; and B sells 2 of his shares to the company, who pay for them from the public stock. After this, A wishes to dispose of the remainder of his part. What proportion of the whole stock does he own?

B owned $\frac{4}{32}$ of the whole stock, and after selling $\frac{3}{32}$ of it, he had $\frac{1}{32}$ part left. The 2 shares of B are divided equally among the other shares; 30 shares are now left, which are equal as before; therefore,

Ans. A owns $\frac{1}{30}$ of the whole.

9. Two persons purchase a house for $4200; the first could pay for the whole, if the second would give him $\frac{1}{3}$ of his money; and the second could pay for the whole, if the first would give him $\frac{1}{5}$ of his money. How much money had each? Multiplying the first condition by 3 gives $12600, the whole money of the second and three times that of the first; taking from this sum $4200, there remains $8400, which is the difference between $\frac{1}{5}$ of the first and three times the first, or $\frac{14}{5}$ of the first. $8400 is $\frac{14}{5}$ of $3000, the money of the first.

Ans. The first had $3000.
The second " 3600.

10. A hare has 50 leaps before a grey-hound, and takes 4 leaps to his 3; but 2 of the grey-hound's leaps are equal to 3 of the hare's. How many leaps must the grey-hound take to overtake the hare?

While the grey-hound takes 1 leap, the hare takes $1\frac{1}{3}$; and while the grey-hound takes 2 leaps, the hare will take $2\frac{2}{3}$. But the grey-hound leaps as far at 2 leaps as the hare does at 3, therefore in 2 leaps he gains $\frac{1}{3}$ of one leap of the hare, or $\frac{1}{6}$ at each leap; hence he will overtake her at 6 times 50 or 300 leaps. Ans. 300 leaps.

11. A person being asked the time of day answered, "the time past from noon is equal to $\frac{5}{17}$ of the time to midnight." What hour of the day was it? Reducing the fractions to a common denominator, $\frac{27}{63}$ of the time past is equal to $\frac{14}{63}$ of the time to come. Therefore $\frac{14}{41}$ of 12 hours will be the time. Ans. 4h. 5min. 51$\frac{9}{41}$ sec.

12. A and B have the same income, A saves $\frac{1}{5}$ of his, but B, by spending £30 a year more than A, at the end of 8 years finds himself £40 in debt. What is their income, and what does each spend per year? In 8 years B finds himself £40 in debt, or he gets in debt £5 a year; he therefore spends £5 more than his income. A spends $\frac{4}{5}$ of his, and B spends £5 more than $\frac{4}{5}$ of his. Hence £25 must be $\frac{1}{5}$ of his income. Ans. £125.

13. A person has two silver cups and only one cover. The first cup weighs 12oz. If the first cup be covered, it weighs twice as much as the second, but if the second cup be covered, it will weigh three times as much as the first? Required the weight of the cover and the second cup. Ans. Cover 20oz. and second cup 16oz.

14. There is a fish whose head is 16 inches long, his tail is as long as his head and half the length of his body, and his body is as long as his head and tail. What is the length of the fish? Ans. 128 inches.

202. Among arithmeticians many other rules are found, which are likewise applications of the Rules of Three;—such as Interest, Equation of Payments, Discount, &c. Those, who have thoroughly understood the principles already established, will have little difficulty in solving questions belonging to such rules.

SIMPLE INTEREST.

Interest is a premium allowed by the borrower to the lender of money. The sum lent is called the *principal*. The *amount* is the sum of the principal and interest. What is called the *rate* of interest, is an allowance of a certain number of pounds, or dollars, &c. for the use of an hundred pounds, dollars, &c. for one year. *Simple* Interest is what is allowed for the principal lent. The following is the rule for computing Simple Interest.

Multiply the principal by the rate and divide the product by 100; *the quotient is the interest for one year.*

Multiply the interest for one year by the given time, and the product will be the answer for that time.

When parts of a year are given, as months or days, compute the interest for months, by taking the fractional parts of a year; and for days, the fractional parts of a month, or by proportions in the Rule of Three.

EXAMPLES FOR PRACTICE

1. What is the interest of £450 for one year, at 5 per cent. per annum? 5 per cent. is $\frac{5}{100}$, hence Ans. £22 10s.

2. What is the interest of £720 for 3 years, at 5 per cent. per annum? Ans. £108.

3. What is the interest of £170 for 1½ years, at 5 per cent. per annum. Ans. £12 15s.

4. What is the interest of $43,25 for 1 year, at 6 per cent? Ans. $2,59,5.

5. What is the interest of 183 dollars, 58 cents for 1 year, at 7 per cent? Ans. Dolls. 12,85.

6. At 6 per cent. for 1 year, what would be the rate per cent. for 13 months? For 14 months?
Ans. $0,065. $0,07.

7. What is the interest of $284,85 for 3 months, at 6 per cent? Ans. $4,27,3.

8. What is the interest of $875,62 for 4 months and 24 days at 6 per cent? Ans. $20,96,6.

9. A note was given on the 7th of March, 1818, for $587; on the 19th of May, 1819, a payment was made of $153; the remainder was paid January 11th, 1820. What was the interest on the note? Ans. $62,87,6.

10. What is the interest of $143,18 for 1 year, 7 months, and 14 days, at 6 per cent? Ans. $13,93.

11. What is the interest of £749 10s. 6d. for 12 days, at 6 per cent? Ans £1 9s. 6d. 3qrs.

NOTE. In computing interest on notes of hand and other instruments, on which indorsements are made, it is usual to calculate the interest on the whole sum up to the time of the first payment, and adding the interest to the principal, to subtract the sum paid from their amount.

Interest is then cast on the sum which remains due, up to the time of the next payment; from the amount of that sum and the interest, which has accrued upon it, subtract the second payment; and so on for every successive payment, to the day of giving up the obligation.

COMPOUND INTEREST.

In general, when a quantity increases, and the rate of increase is proportional to that quantity, the resulting products will be quantities in geometrical proportion. Hence, in *Compound* Interest, if a given sum of money be supposed to produce, in a limited time, a certain sum; in double that time, the first sum will have produced the like sum again; and the sum, produced in the first space of time, will also have produced, in the second space, another sum proportional to itself; and so on.

As in simple interest, the first sum is called the *principal;* the sum produced, the *interest;* and the constant ratio of the principal to the interest, per annum, is called the *rate.* Thus, five per cent. per annum shows, that the rate of interest for a year is the twentieth part of the principal, which is also called 5 for 100; because 5 is the twentieth part of 100. On this principle, the first sum will be augmented at the end of the year one twentieth part. At the end of two years, it will be augmented again in the same proportion, that is, in the ratio of $\frac{21}{20}$, multiplied by $\frac{21}{20}$; at the end of three years, in the ratio of $\frac{21}{20}$ multiplied three times by itself; and so on. In this manner it is found, that a sum is doubled in about fifteen years; and in fifty-three years, becomes ten times greater. Then, reciprocally, if a sum actually paid is doubled in fifteen years, it is plain, that a sum, which does not become due until the expiration of fifteen years, is worth but half of its nominal value. This is what is called the *present worth* of a sum payable at the expiration of a given time; and to find this value, we have only to divide the sum promised, as many times by the fraction $\frac{21}{20}$, (or rather to multiply it by the fraction $\frac{20}{21}$) as there are years before it becomes due. Thus it is found, that the present worth of a sum payable in fifty-three years is only one tenth of its nominal value.

The rule for computing compound interest is as follows:

Find the amount of the given principal to the time of the first payment, by simple interest. Consider this amount as the principal, on which calculate the interest as before, to the time of the second payment; and so on through all the payments, considering the last amount as the principal for the next payment.

EXAMPLES FOR PRACTICE.

1. The amount of £320 10s. for 4 years, at 5 per cent. compound interest, is required.

£320 10s.	1st year's principal.
16 0 6d.	1st year's interest.
336 10s. 6d.	2d year's principal.
16 16 6¼.	2d year's interest.
353 7s. 0¼d.	3d year's principal.
17 13 4	3d year's interest.
371 0s. 4¼d.	4th year's principal.
18 11 0	4th year's interest.
£389 11s. 4¼d. Ans.	

2. What is the compound interest of £410, payable in 2½ years, at 4½ per cent. per annum; interest payable half yearly? Ans. £48 3s. 6d. 1 qr.

3. What is the amount of £721 for 21 years, at 4 per cent. per annum? £1642 19s. 10d.

4. What will $2,12 amount to in 4 years, at 12 per cent. per annum, compound interest? Ans. $3,33 &c.

5. What is the compound interest of doll.33,33 for two years, at 10 per cent.? Ans. doll.6,99,9, &c.

6. What is the compound interest of £760 10s. for 4 years, at 6 per cent. per annum? Ans. £199 12s. 2d.

7. What is the compound interest of doll.500 for 4 years, at 6 per cent. per annum? Ans. doll.131,23¾.

8. How much will £400 amount to in 4 years, at 6 per cent. per annum? Ans. £504 19s. $9\frac{1}{4}$d.

9. How much will doll.680 amount to in 4 years at 6 per cent. compound interest? Ans. doll.858,48,3.

10. What is the compound interest of doll.840 for 3 years, at 5 per cent. per annum? Ans. doll.132,48,5.

EQUATION OF PAYMENTS.

The following is the rule commonly used for performing this operation.

Multiply each payment by the time, at which it becomes due; divide the sum of these products by the sum of the payments, and the quotient will be the time sought.

This rule, though sufficiently accurate for business, is not strictly true; for although by keeping a debt unpaid after it is due, the interest of it is gained for that time; yet by paying a debt before it is due, the payer does not lose the interest for that time, but the discount, which is less than the interest.

EXAMPLES FOR PRACTICE.

1. A owes B £190 to be paid as follows,—£50 in 6 months, £60 in 7 months, and £80 in 10 months; what is the equated time at which to pay the whole?

$$\begin{array}{rr} 50\times 6=300 & 50 \\ 60\times 7=420 & 60 \\ 80\times 10=800 & 80 \\ \hline 1520 & 190 \\ \hline & 8 \text{ months. Ans.} \end{array}$$

2. A owes B doll.7 to be paid in 3 months, and doll.5 to be paid in 8 months; what is the equated time for the payment of the whole? Ans. $5\frac{1}{12}$ months.

3. G owes H doll.1000, of which doll.200 is to be paid immediately, doll.400 in 5 months, and the remainder in 15 months; if one payment is made of the whole, in what time ought it to be made? Ans. 8 months.

4. A debt is to be paid as follows,— $\frac{1}{4}$ in 2 months, $\frac{1}{8}$ in 3 months, $\frac{1}{8}$ in 4 months, $\frac{1}{8}$ in 5 months, and the remainder in 7 months; what is the equated time for the payment of the whole? Ans. 4 months and 18 days.

5. G owes K £420, due 6 months hence; it is agreed that £60 shall be paid now, and that the remainder continue unpaid a longer time than 6 months; when must it be paid? Ans. in 7 months.

DISCOUNT.

By this rule is found the difference in value between a sum due at the expiration of a given time, and its present worth. The following is the method of operation.

Find the amount of £100, or $100, for the given rate and time; and make it the first term of a proportion, which shall have £100, or $100, for its second term, and the given sum or debt for the third term. The fourth term of this proportion, when found, will be the present worth; from which, subtracting the given sum, the remainder will be the discount required.

EXAMPLES FOR PRACTICE.

1. What is the discount of £573 15s. due 3 years hence, at 4½ per cent? Ans. £68 4s. 10¼d.

2. What is the discount of $750 payable in one year? $22,50.

3. What is the present worth of $590, due in 3 years, discounting at the rate of 6 per cent.? Ans. $500.

4. What is the difference between the interest of $1204, at 5 per cent. per annum, for 8 years; and the discount of the same sum for the same time and rate per cent.? Ans. $137,60.

5. What is the present worth of $43,67, due in 19 months; discounting at 5 per cent. per annum? Ans. $399,07.

BARTER.

This rule, which is an application of the rules of proportion, is used by merchants for proportioning their goods in such manner, that, on making an exchange of commodities, neither party may sustain a loss. The method of operation is as follows:

Find the value of the commodity whose quantity is given, and then find what quantity of the other, at the rate proposed, may be had for the same money; and it will be the answer required.

EXAMPLES FOR PRACTICE.

1. How much sugar at 9d. per lb. must be given in barter for 492lb. of rice, at 3d. per lb. Ans. 164lbs.

2. How much wheat at $1,25 per bushel is equal in value to 50 bushels of rye at $0,70 per bushel.
Ans. 28 bushels.

3. A and B barter; A has 320lb. of chocolate, at 4s. 6d. per lb. for which B is to give him £30 in money, and the rest in cotton at 8d. per lb. How much cotton is B to give A? Ans. 1260lbs.

4. Bought a sloop of 70 tons for $16 per ton; paid $500 in cash, 350 gallons of molasses at $0,64 per gallon, and the balance in New-England rum at $0,74 per gallon: how many gallons did it amount to?
Ans. $535\frac{5}{37}$ gallons.

5. R gave 189 yards of linen, at 6s. 8d. per yard, to C for 42 yards of cloth; what was the cloth per yard?
Ans. 30s.

LOSS AND GAIN.

By this rule merchants so estimate their goods in buying and selling, as to know what they gain or lose. *As the gains or losses are in proportion to the quantities of goods, questions in this rule are solved by applications of the Rules of Three.*

EXAMPLES FOR PRACTICE.

1. Bought 650lbs. of sugar, at 10 cents per lb. and sold it at 12 cents per lb.; how much did I gain? Ans. $13.

2. A merchant bought 1300lb. of coffee, at 22 cents per lb. and was afterwards obliged to sell it for 20 cents per lb.; how much did he lose? Ans. $26.

3. If a trader gain 1½d. per shilling on his goods, how much does he gain per cent.? Ans. 12½.

4. A parcel of goods were bought for £18 and immediately sold for £25, at 4 months credit; what was gained per cent. per annum?

5. A repeating watch was sold for $175, on which 17 per cent. was lost, whereas 20 per cent. ought to have been gained; what was the watch sold for under its value?

100—17 : 175 :: 17+20 : $78,1, &c.

Ans. $78,1, &c.

COMMISSION AND BROKERAGE.

Commission and Brokerage are allowances made at a certain rate per cent. to Factors and Brokers, for their services. *The method of operation in these rules is the same as in Simple Interest.*

EXAMPLES FOR PRACTICE.

1. What is the commission on £500 13s. 6d., at 3½ per cent.? Ans. £17 10s. 5½d.

2. What is the brokerage on £879 18s. at ⅜ per cent.? Ans. £3 5s. 11¾d.

3. What is the commission on $2176,50 at 2½ per cent.? Ans. $54,41,2.

4. What is the brokerage on $1298,53 at ⅜ per cent.? Ans. $4,86,9.

5. A factor has $3690, with which he is to purchase iron, reserving from it his commission of 2½ per cent.; how much iron can he purchase at $95 per ton? Ans. 37 tons. 17cwt. 3qrs. 16 4/19lbs.

INSURANCE.

Insurance is a premium, at a certain rate per cent., given to certain persons or offices, on condition of being indemnified for losses by fire, storms, &c. *The method of operation is the same as for Interest.*

EXAMPLES FOR PRACTICE.

1. What is the premium for insuring \$4500, at 25 per cent.? Ans. \$1125.

2. What is the premium for insuring \$4000, at $7\frac{5}{8}$ per cent.? Ans. \$305.

3. What is the insurance of £900, at $10\frac{3}{4}$ per cent.? Ans. £96 15s.

4. What is the premium for insuring \$1650, at $15\frac{1}{2}$ per cent.? Ans. \$255,75.

5. What is the insurance of £1200, at $7\frac{5}{8}$ per cent.? Ans. £91 10s.

TARE AND TRET.

These are practical rules for deducting certain allowances, made by merchants in selling their goods by weight. *Gross weight* is the whole weight of any goods, together with the box, barrel, &c. containing them. *Tare* is a certain allowance made to the buyer for the weight of the box, barrel, &c., and is either so much per box, &c. or so much per cwt. gross. *Tret* is an allowance of 4lb. for every 104, for waste, dust, &c. *Cloff* is an allowance of 2lb. in every 3cwt. *Suttle* is what remains after a part of the allowance is deducted from the gross weight. *Net weight* is what is left after all allowances are made.

CASE I.

When the tare is a certain weight per box, barrel, &c.

Multiply the number of boxes, barrels, &c. by the tare, subtract the product from the gross, and the remainder will be the net weight.

1. In 241 barrels of figs, each weighing 3cwt. 19lb. gross; tare 10lb. per barrel; how many pounds net, and what is their value at 15¾ cents per lb.?

Ans. { 22413lbs. net.
$3530,04,7, &c. value.

2. What is the net weight of 4 chests of tea, weighing gross, 96lb. 97lb. 101lb. and 103lb.; tare 20lb. per chest?

Ans. 317lb.

CASE II.

When the tare is a certain weight per cwt.

Divide the gross by the aliquot, or fractional part, which the tare makes of a cwt.; subtract the quotient from the gross, and the remainder will be the net weight.

1. At $10,50 per cwt. what is the value of 12hhds. of sugar, the gross weight being 9[illegible]cwt. 1qr. 19lb.; tare 18lb. per cwt.?

Ans. { $823,26,5⅝.
78cwt. 1qr. 17½lb. net.

2. What is the net weight of 7 barrels of potash, each weighing 402lb. gross, tare 10lb. per cwt.?

Ans. 2562lbs. 12oz.

CASE III.

When tret is allowed with the tare.

Find the tare as before directed, and divide the suttle weight by 26, the aliquot part that 4lb. *makes on every* 104; *the quotient will be the tret, which subtracted from the suttle, the remainder will be the net weight.*

1. What is the net weight of a hhd. of tobacco, weighing 14cwt. 3qrs. 27lb. gross; tare 16lb. per cwt.; tret 4lb. per 104lb.; how many pounds net, and what is the value at 6½ cents per lb.?

Ans. { 1383lbs. 14oz. net.
$89,95,1⅞ value.

2. In 7 casks of prunes, each weighing 3cwt. 1qr. 5lb. gross; tare 17½lb. per cwt. and tret as usual, how much net?

Ans. 18cwt. 2qrs. 25lb.

CASE IV.

When tare, tret, and cloff, are allowed.

Deduct the tare and tret as before, divide the suttle by 168, *and the quotient is the cloff; subtract the cloff from the suttle, and the remainder is the net weight.*

1. What is the net weight of 15cwt. 3qrs. 20lb. gross; tare 7lb. per cwt.; tret and cloff as usual?

Ans. 14cwt. 1qr. 2lb. 10oz.

2. In 19 chests of sugar, each weighing 13cwt. 1qr. 17lb. gross; tare 13lb. per cwt.; tret and cloff as usual; how much net, and what is the value at $5\frac{3}{4}$d. per lb.?

Ans. { 215cwt. 16lb. net.
£577 6s. value.

GUAGING.

This rule is used for taking the dimensions of a cask in inches, to find its contents in gallons. The method of operation is as follows.

Add two thirds of the difference between the head and bung diameters, for the mean diameter. Square the mean diameter, and multiply its square by the length of the cask; divide this product by 294 *for wine, or by* 359 *for ale measure, and the quotient will be the answer in gallons.*

EXAMPLE.

How many ale or beer gallons will a cask hold, whose bung diameter is 31 inches, head diameter 25 inches, and length, 36 inches? Ans. 84gall. $1\frac{1}{3}\frac{2}{5}\frac{1}{9}$qts.

MISCELLANEOUS EXAMPLES IN RULES UNDER ARTICLE (202.)

1. What is the interest of $57,78 for one year, 4 months, and 17 days, at 4 per cent. per annum?

Ans. $3,19.

2. What is the amount of $298,59 from May 19th, 1797, to August 11th, 1798, at 8 per cent. per annum?

Ans. $327,98,4.

3. What is the interest of $7086 for 39 days, at 6 per cent. per annum? Ans. $45,43.

4. What is the compound interest of $56,75 for 11 years? Ans. $50,97.

5. What is the amount of $236 at compound interest, for 4 years, 7 months, and 6 days? Ans. $308,66,9.

6. What is the compound interest of $1256 for 15 years at 6 per cent. per annum? Ans. $1754,06,6.

7. C owes D $100 on demand; $100 in one year; $100 in two years; and $100 in three years; what is the equated time to pay all at once? Ans. 1½ years.

8. A merchant purchased goods to the value of $4000, to be paid as follows;—$1200 in 4 months; $800 in six months, and the remainder in 12 months; but afterwards agreed to pay the whole at a time; when must the debt be paid? Ans. in 8 months and 12 days.

9. A owes B $840 to be paid as follows; ⅓ in 3 months, ¼ in 4 months, and the remainder in 6 months; at what time must the whole be paid at once?
Ans. 4½ months.

10. What is the present worth of $874 at 6 per cent. due 4 years hence? Ans. $704,83, &c.

11. What is the discount of $846 for six months, at 6 per cent.? Ans, $24,64.

12. What is the discount upon $560 due 9 months hence, at 8 per cent. per annum? Ans. $31,69, &c.

13. A has raisins at 25 cents per lb. ready money, but in barter will have 30 cents per lb.; B has tea at $1,75 per lb. ready money; what ought B to rate his tea at in barter, and what quantity of tea must be given for 20 boxes of A's raisins, each weighing 35lb.?

Ans. { $2,10 B's barter price.
{ 100lb. quantity of tea.

14. Two merchants barter; A has 120cwt. of sugar at $12 per cwt.; B has 256 yards of broadcloth at $5 per yard; who must receive the difference, and how much?
Ans. $160 in A's favour.

15. A exchanges with B 40lb. of indigo at $1 per lb. ready money, and $1,25 in barter, but is willing to lose 10 per cent. to have ⅓ ready money; what is the ready money price of 1 yard of cloth delivered by B at $3,50,

to equal A's bartering price; and how many yards were delivered?

Ans. { $3,11,1⅕ money price of 1 yard.
{ 12$\frac{5}{7}$ yards delivered by B.

16. Bought 37 gallons of brandy at $1,10 per gallon, and sold it for $40; what was gained or lost per cent.?

Ans. $1,71,9 lost.

17. Hats were bought at 4s. each, and sold again at 4s. 9d.; what was the profit in laying out £100 at that rate? Ans. £18 15s.

18. Wheat is bought at $1,25 and sold at $1,50 per bushel; what is the gain per cent.? Ans. 20 per cent.

19. What is the premium on $4844, at 3½ per cent.?

Ans. $169,54.

20. What is the premium on a ship and cargo valued at $19330, at 3¼ per cent. from Boston to London, and 2¾ per cent. from London to Boston? Ans. $1159,80.

21. What is the insurance of an East India ship and cargo, valued at £14813 15s., at 15¾ per cent.?

Ans. £2333 3s. 3¾d.

22. What must be allowed to a correspondent for selling goods to the amount of $48450, at 2⅛ per cent.?

Ans. $1029,56,2½.

23. What is due to a factor for selling goods, valued at $1834, at 2½ per cent? Ans. $45,85.

24. What is the commission on the purchase of goods, the invoice of which amounts to $870, at 2¾ per cent?

Ans. $23,92,5.

25. A has in his possession a hhd. of sugar, weighing gross 9cwt. 3qrs. and owned equally between him and B. How much sugar must he weigh out to B. allowing tare to be 12lb. per cwt? Ans. 4cwt. 1qr. 11½lb.

ALLIGATION.

203. The questions, which pertain to this rule, are of two kinds. In one, it is required to find the mean value of many sorts of things, of which the number and particular value of each are known. In the other, it is required to find the quantities of each kind of things, which enter into one or many mixtures; when we know the

price or value of each kind, and the price or total value of each mixture. Questions of the second sort are usually solved by Algebra, although Arithmetic may serve for the purpose, as will be shown hereafter. The following is the rule for questions of the first kind:

Multiply the value of each kind of things by the number of things of that sort; add all the products, and divide their sum by the total number of things of all the kinds.

EXAMPLE.

A man employed 200 labourers; 50 of whom he paid at the rate of 40d. a day; 70 at the rate of 30d.; 50 at the rate of 25d.; and 30 at the rate of 20d. What is the average price each labourer receives a day?

50 labourers, at 40d. a day, receive	2000d.
70 at 30d.	2100d.
50 at 25d.	1250d.
30 at 20d.	600d.
	5950d.

The cost of 200 labourers is then 5950d. a day; and dividing this sum by 200, the quotient, 29d. $3\frac{16}{20}$qrs. shows the average price each labourer receives.

EXAMPLES FOR PRACTICE.

1. A refiner has 5lb. of silver bullion, of 8oz. fine; 10lb. of 7oz. fine; and 15lb. 6oz. fine; on melting them all together, what would be the fineness of 1lb. of this mass? Ans. 6oz. 13dwts. 8grs. fine.

2. A farmer mixed 19 bushels of wheat, at 6s. per bushel; 40 bushels of rye, at 4s. per bushel; and 12 bushels of barley, at 3s. per bushel; what was a bushel of this mixture worth? Ans. 4s. 4d. $1\frac{4}{7}$qrs.

3. If 16 gallons of brandy at $1,25 per gallon, and 4 gallons of water be mixed with 40 gallons of wine, at $3 per gallon; what will the mixture be worth per gallon? Ans. $-,33$\frac{1}{3}$.

4. A grocer mixed 10 gallons of water with a hhd. of rum, worth $0,87 per gallon; what was the mixture worth per gallon? Ans. $0,75$\frac{6}{73}$.

5. A grocer mixed 43 gallons of wine at $1,25 per gallon, with 87 gallons worth $1,60 per gallon. What was the mixture worth per gallon? Ans. $1,48,4$\frac{3}{13}$.

6. A composition is made of 5lb. of tea, at 7s. per pound; 9lbs. at 8s. 6d. per pound; and 14½lbs. at 6s. 10½d. per pound; what is a pound of the mixture worth?
Ans. 7s. 4¾d &c.

ALLIGATION ALTERNATE.

This rule is the reverse of Alligation, and may be proved by it. The method of operation is as follows:

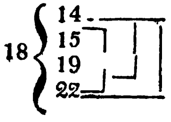
"*Write the rates of the simples in a column under each other. Connect with a continued line the rate of each simple, which is less than that of the compound, with one or any number of those, that are greater than the compound; and each greater rate with one or any number less.*

Write the difference between the mixture rate and that of each of the simples opposite the rates, with which they are respectively linked. Then if only one difference stand against any rate, it will be the quantity belonging to that rate; but if there be several, their sum will be the quantity."

EXAMPLES.

A merchant would mix wines at 14s. 19s. 15s. and 22s. per gallon, and have the mixture worth 18s. per gal. What quantity of each must be taken?

The quantities are written as follows;



1+4=5 at 14s.
1 =1 at 15s.
4+3=7 at 19s.
4 =4 at 22s.
Ans.

2 It is required to mix brandy at 80 cents, wine at 70 cents, cider at 10 cents, and water together, so that the mixture may be worth 50 cents per gallon. Ans. 9 gallons of brandy, 9 of wine, 5 of cider, and 5 of water.

CASE II.

When the whole composition is limited to a certain quantity, "*find an answer as before by linking; then say, as the sum of the quantities, or differences thus determined, is to the given quantity, so is each ingredient, found by linking, to the required quantity of each.*"

EXAMPLES.

1. How much water, considered as of no value, must be mixed with wine worth 3s. per gallon, so as to fill a vessel of 100 gallons, and that a gallon may be afforded at 2s. 6d.?

$$30\left\{\begin{array}{r}0\\36\end{array}\right]\begin{array}{r}6\\30\\\hline 36\end{array}$$

36 : 100 :: 6 : 83$\frac{12}{36}$ galls. of wine, 36 : 100 :: 30 : 16$\frac{24}{36}$ galls. of water. Ans.

2. How much gold of 15, of 17, of 18, and of 22 carats fine, must be mixed together to form a composition of 40 ounces, 20 carats fine?

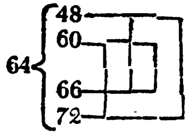
Ans. 5oz. of 15, 17, and 18, and 25oz. of 22.

CASE III.

When one of the ingredients is limited to a certain quantity, *find the difference between each price and the mean rate as before; then, as the difference of that simple whose quantity is given, is to the other differences severally, so is the quantity given to the several quantities required.*

EXAMPLES.

1. How much wine at 5s., 5s. 6d., and 6s. the gallon, must be mixed with 3 gallons, at 4s. per gallon, so that the mixture may be worth 5s. 4d. per gallon?



8+2=10
8+2=10
16+4=20
16+4=20

10 : 10 :: 3 : 3 gallons at 5s.
10 : 20 :: 3 : 6 gallons at 5s. 6d. } Ans.
10 : 20 :: 3 : 6 gallons at 6s.

2. How much gold of 15, of 17, and of 22 carats fine, must be mixed with 5oz. of 18 carats fine, so that the composition may be 20 carats fine?

Ans. 5oz. of 15 carats fine, 5oz. of 17, and 25 of 22.

EXAMPLES FOR PRACTICE.

1. A grocer would mix sugars at 10, 13, and 16 cents per pound. What quantity of each must he take to make a mixture worth 12 cents per lb.?

Ans. 5lb. at 10 cents, 2lb. at 13cts. and 2lb. at 16cts.

2. A farmer had 10 bushels of wheat, worth 8s. per bushel, which he wished to mix with corn, at 3s. per bushel, so that the mixture might be worth 5s. per bushel. How many bushels of corn must he use?

Ans. 1 of wheat to 1½ of corn, or 10 of wheat to 15 of corn.

3. A grocer has rum worth 75 cents per gallon; how many parts water must he put in, that he may afford to sell the mixture at 25 cents per gallon?

Ans. 2 parts water to 13 rum.

4. A grocer has two sorts of tea, one at 75 cents per pound, and the other at $1,10 per lb? How must they be mixed, that the mixture may be afforded at $1,00 per lb.? Ans. 2 parts of the cheaper to 5 of the dearer.

ARITHMETICAL PROGRESSION.

204. Arithmetical progression is a series of terms, in which each exceeds the term that precedes it, or is exceeded by it, by the same quantity.

The following series is an arithmetical progression; ÷ 1 · 4 · 7 · 10 · 13 · 16 · 19 · 22 · 25 · 28 · 31 · 35 · &c.:—because each term exceeds that which precedes it, by the same quantity 3.

The three points separated by a bar, which stand at the beginning of the series, denote that, in enunciating the progression, each term except the first and last, must be repeated in this manner, 1 *is to* 4 *as* 4 *is to* 7, *as* 7 *is to* 10, &c.

The progression is called *increasing* or *decreasing* according as the terms of the series go on augmenting or diminishing. But as the properties of both are the same, (by changing the words *plus*, or *addition*, into *minus* or *subtraction*), we shall consider the progression only as increasing.

205. It is plain, after the definition of arithmetical progression, that, with the first term and common difference or ratio of the progression, all other terms of the series may be formed by successively adding this ratio. Hence, the second term is composed of the first plus the ratio. The third is composed of the second, plus the ratio; consequently, of the first plus twice the ratio. The fourth is composed of the third, plus the ratio; and consequently, of the first, plus three times the ratio; and so on.

206. Hence, in general, *any term whatever of an arithmetical progression is composed of the first, plus as many times the ratio as there are terms in the series preceding it.*

207. Then, the first term being cipher, every other term of the progression will be equal to as many times the ratio, as there are preceding terms in the series.

208. This principle may have the two following applications.

1st. It serves for finding any term of a progression, without requiring a calculation of those, that precede it.

For example, if it be required to know what is the 100th term of this progression.

$\div 4 \cdot 9 \cdot 14 \cdot 19 \cdot 24$, &c.

Since the hundredth is the term sought, there must be 99 terms before it; it is then composed of the first term 4, and 99 times the ratio 5; consequently, the term sought is 4 plus 495=499.

209. 2d. The same principle is used for connecting any two numbers, by a series of other numbers consisting of any number of terms, in such manner, that the whole shall form an arithmetical progression. This operation is called the *insertion*, between two given numbers, of many *arithmetical mean proportionals*, or simply, many *arithmetical means*.

For example, it is required to connect 1 and 7, by five numbers, which form an arithmetical progression with 1 and 7; these numbers are 2, 3, 4, 5, 6. But as it is not always easy to perceive immediately what these numbers must be, we will show how they may be known by means of the principle already explained. For this purpose, we have only to find the ratio between the terms of the progression. And as the greater of the two given numbers is the last term of the progression, it must be composed of the first, (that is the smaller of these two numbers,) plus as many times the ratio as there are terms to precede it; then, if from the greater of the two numbers we subtract the less, the remainder will be composed of as many times the ratio, as there ought to be terms in the series, preceding the greatest. Or, the remainder is the product of the multiplication of this ratio by the number of terms, which precede the greatest. Then (174) if we divide the remainder by the number of terms, which ought to precede the greatest, the quotient will be this ratio. But the number of terms, which are to precede the greatest, is greater by unity than the number of means, which must be inserted between the two.—Then, *to insert, between two given numbers, any given number of arithmetical means, it is necessary to subtract the less from the greater of the two numbers, and to divide the remainder by the number of means augmented by unity.* The quotient will be the difference or ratio, which must exist between the terms of the progression. For example, if it be required to insert 8 arithmetical means, between 4 and 11, we subtract 4 from 11, and divide the remainder 7 by 9, the number of terms augmented by unity; the quotient $\frac{7}{9}$ is the difference, or ratio required between the terms of the progression; which, consequently, will be

$$\div 4 \cdot 4\tfrac{7}{9} \cdot 5\tfrac{5}{9} \cdot 6\tfrac{3}{9} \cdot 7\tfrac{1}{9} \cdot 7\tfrac{8}{9} \cdot 8\tfrac{6}{9} \cdot 9\tfrac{4}{9} \cdot 10\tfrac{2}{9} \cdot 11.$$

In like manner, if the nine arithmetical means, between 0 and 1, are required; after subtracting 0 from 1, there remains 1, which, divided by 10, the number of means augmented by unity, gives $\frac{1}{10}$, or 0,1 for the ratio. The progression, therefore, will be

$$\div 0 \cdot 0{,}1 \cdot 0{,}2 \cdot 0{,}3 \cdot 0{,}4 \cdot 0{,}5 \cdot 0{,}6 \cdot 0{,}7 \cdot 0{,}8 \cdot 0{,}9 \cdot 1.$$

210. Hence it is plain, that if two numbers be adjacent, any required number of arithmetical means may be inserted between them.

The subject of arithmetical progressions will be resumed again when treating of logarithms. The principles already explained will sufficiently illustrate the rules in the two following cases.

CASE I.

When the first term, common ratio, and number of terms are given, to find the last term, and sum of all the terms, the operation is as follows.

Multiply the number of terms, less 1, *by the common difference; to the product add the first term, and their sum will be the last term. Multiply the sum of the two extremes by the number of terms, and half the product will be the sum of all the terms.*

EXAMPLES.

1. The first term of a certain series in arithmetical progression is 2, the ratio between the terms is 2, and the number of terms 15; what is the last term, and the sum of all the terms?

Ans. { 15 number of terms.
240 sum of all the terms.

2. How many strokes does a Venice clock strike in a day, going to 24 o'clock? Ans. 300.

3. 16 persons gave charity to a poor man; the first gave 5d., the second 9d. and so on in arithmetical progression; how much did the last person give; and what sum did the man receive?

Ans. { The last man gave 5s. 5d.
The sum received £2 6s. 8d.

4. If 100 stones be laid two yards distant from each other in a right line, and a basket placed two yards from the first stone; what distance must a person travel to gather them singly into the basket?

Ans. 11miles, 3fur. 180yds.

CASE II.

When the two extremes and number of terms are given, to find the common ratio or difference. *Divide the difference of the extremes by the number of terms less* 1; *the quotient will be the common ratio.*

EXAMPLES.

1. 20 and 60 are the two extremes of a series in arithmetical progression, and 21 is the number of terms; what is the common difference? Ans. 2.

2. A man had 12 sons whose several ages differed alike; the eldest was 49, the youngest 5 years old; what was the common difference of their ages?

Ans. 4 years.

3. A debt is to be paid at 16 different payments in arithmetical progression; the first payment to be £14, and the last £100; what is the common difference, each payment, and the whole debt?

Ans. { Common difference £5 14s. 8d.
First payment £14. Second £19 4s. 8d.
Third £25 9s. 4d. &c.

4. If a man travel the first day 7 miles, and the last day 51 miles, increasing his journey 4 miles each day; how many days will he travel, and how far?

Ans. 12 days, and 348 miles.

GEOMETRICAL PROGRESSION.

211. Geometrical progression is a series of terms, in which each contains that which precedes it, or is contained in it, the same number of times.—The following series

÷ 3 : 6 : 12 : 24 : 48 : 96 : 192, is a geometrical progression; because each term contains that which precedes it, the same number of times,—that is, twice. This number of times is called the *ratio* of the progression. The three points and bar have the same signification as in arithmetical progression.

The progression is called *increasing* or *decreasing*, according as the terms go on augmenting or diminishing. But we may always consider a geometrical progression as increasing, because the properties are made the same in both cases by changing the word *multiplication* into *division*, and *contain*, into *be contained.*

As the second term contains the first as many times as there are units in the ratio, it must be composed of the first term multiplied by the ratio. And as the third term contains the second as many times as there are units in the ratio, it must be composed of the first multiplied by the square or second power of the ratio. And as the fourth term contains the third as many times as there are units in the ratio, it must be composed of the first, multiplied by the cube, or third power of the ratio. For example, in the *preceding progression*, 6 is composed of the first term 3 multiplied by the ratio 2; 12 is composed of the first term 3 multiplied by 4, the square of the ratio 2; 24 is composed of the first term 3 multiplied by 8, the cube of he ratio 2.

212. By continuing the same mode of reasoning, it is seen that, *any term whatever of a geometrical progression is composed of the first term multiplied by the ratio, raised to a power indicated by the number of terms, which precede that term.*

Then, if the first term of a progression be unity, each other term will be formed of the ratio raised to a power, indicated by the number of terms which precede it; because multiplication by the first term, which is unity, does not augment the product.

To raise a number to a given power, to the seventh for instance, it is necessary, according to the idea which has been given of powers, to multiply the number by itself six times consecutively; thus, to raise 2 to the seventh power, we say 2 times 2 make 4, 2 times 4 are 8, two times 8 make 16, two times 16 make 32, twice 32 make 64, twice 64 are 128, which is the seventh power of 2. But we can abridge the operation by various methods; for example, we first square 2, which makes 4; then cube 4, which makes 64, and then multiply 64 by 2. But it is of little consequence in what manner the operation is performed, provided 2 is made 7 times factor in the product.

213. The principle explained in article (212) for the formation of any term of a progression, and the remark made upon it, may serve for finding any given term in a progression, without requiring a calculation of the terms that precede it. For example, if the twelfth term be required of this progression;

∺ 3 : 6 : 12 : 24, &c.

since we know (212), that the twelfth term must be composed of the first multiplied by the ratio raised to a power indicated by the number of terms, which precede the twelfth; it is plain, that, to form this term, we must multiply 3 by the eleventh power of the ratio 2. To form the eleventh power we cube the number 2 which gives 8; we then find the cube of 8, which is 512, the ninth power of 2; we then multiply 512, the ninth power, by 4, the second power of 2, and have 2048 for the required eleventh power of 2. Then multiplying 2048 by 3, we have 6144 for the twelfth term of the progression.

214. The same principle may be applied for finding, between two given numbers, any required number of geometrical mean proportionals. If three geometrical means are required between 4 and 64; by a little attention, we find, that these terms are 8, 16, 32. And it is plain, that ∺ 4 : 8 : 16 : 32 : 64, form a geometrical progression. But if the same number of means had been required between other numbers than 4 and 64; or if a greater or less number of geometrical means had been required between those same numbers, it might have been attended with some difficulty to find them. But by means of the principle under consideration, this difficulty may

be removed. For the question becomes to find the ratio between the terms of the progression; because, when the ratio is found, the terms may be easily formed by successive multiplications by that ratio. For example, if it be required to find nine geometrical means between 2 and 2048; 2048 will be the last term of a geometrical progression, which commences with 2, and which must have nine terms between the first and the last. The last term 2048 is then composed of the first term 2, multiplied by the ratio, raised to a power indicated by the number of terms, which precede 2048. Then, if we divide 2048 by the first term the quotient will be the ratio raised to a power indicated by the number of terms, which must precede 2048; and then by extracting the root of this power, we obtain the ratio. But the power in question is the tenth; because, there being nine terms between 2 and 2048, there must necessarily be ten which precede 2048. The tenth root, then, must be extracted from the quotient of the greater number 2048, divided by the less number 2.

215. As this reasoning applies to all cases, it may be inferred in general, that, *to insert between two given numbers any required number of geometrical mean proportionals, the greater of the two numbers must be divided by the less, and the root extracted from the quotient, to a degree indicated by the number of means augmented by unity.* Thus, recurring to our example, we divide 2048 by 2, and find 1024 for quotient; from this quotient we extract the tenth root, which is 2; the ratio, then, is 2. Now to form the required means, we multiply the first term 2, continually by the ratio 2; and after having formed nine means, we arrive at the number 2048, as is seen in the following series:—

$$\div\!\!\div\ 2 : 4 : 8 : 16 : 32 : 64 : 128 : 256 : 512 : 1024 : 2048.$$

In like manner, if it be required to find four geometrical means between 6 & 48, we divide 48 by 6, and extract the fifth root from the quotient 8; but as the fifth root of 8 is not an exact number, we cannot exactly assign in numbers the four geometrical means between 6 and 48; this root, however, may be approximated to any required degree of exactness, by a method similar to that explained for the square and cube roots. Hence we may conclude in general, that between any two numbers whatever, any required number of geometrical means may always be found, either exactly, or to a sufficient approximation.

EXAMPLES FOR PRACTICE IN GEOMETRICAL PROGRESSION.

1. Bought 30 bushels of wheat, at 2d. for the first bushel, 4d. for the second, 8d. for the third, &c.; what does the whole amount to? Ans. £8947848 10s. 6d.

2. What sum would purchase a horse with four shoes, and 8 nails in each shoe; at one farthing for the first nail, and two for the second, a penny for the third and doubling to the last? Ans. £4473924 5s. 3¼d.

3. A gentleman married his daughter on new year's day, and gave her one dollar towards her portion, promising to double it on the first day of every month for one year; what was her portion? Ans. $4095.

4. A man dying left 9 sons, to whom and to his executor, he bequeathed his estate in the following manner;—to the executor £50; to his youngest son as much more; and the portion of each son was to exceed that of the next younger by as much more; what was the eldest son's portion? Ans. £25600.

MISCELLANEOUS EXAMPLES.

1. What number is that from which $\frac{3}{7}$ being taken, the remainder will be $\frac{1}{5}$? Ans. $\frac{22}{35}$.

2. What number is that, of which $19\frac{3}{13}$ is $\frac{5}{7}$?

Ans. $19\frac{3}{13}=\frac{250}{13}$; then, as $\frac{5}{1} : \frac{250}{13} :: \frac{7}{1} : 26\frac{12}{13}$, the number required.

3. In an orchard of fruit trees, $\frac{1}{2}$ bear apples, $\frac{1}{4}$ pears, $\frac{1}{6}$ plums; 60 trees bear peaches and 40 cherries. How many trees are there in the orchard. Ans. $\frac{1}{2}+\frac{1}{4}+\frac{1}{6}=\frac{11}{12}$ and $\frac{12}{12}-\frac{11}{12}=\frac{1}{12}$; then, $\frac{1}{12} : \frac{60+40}{1} :: \frac{12}{12} : 1200$, the number of trees.

4. If $\frac{5}{7}$ of $\frac{3}{8}$ of $\frac{4}{5}$ of a ship be worth $\frac{2}{9}$ of $\frac{7}{8}$ of $\frac{12}{13}$ of the cargo, valued at £1000; what is the value of both ship and cargo? Ans. The value of the ship and cargo is £1837 12s. $1\frac{25}{39}$d.

5. A younger brother received $3125,92cts. which was $\frac{7}{12}$ of his elder brother's fortune; and $5\frac{3}{8}$ times the elder's money was $1\frac{2}{3}$ of the value of their father's estate. How much was their father worth?

Ans. $17281,87,2.

6. A gentleman left a fortune to his son, $\frac{5}{16}$ of which he spent in 3 months, $\frac{3}{4}$ of $\frac{5}{6}$ of the remainder lasted him 9 months longer, when he had £537 left. What was the fortune he inherited? Ans. $\frac{16}{16}$=whole legacy, $\frac{16}{16}-\frac{5}{16}=\frac{11}{16}$ left at 3 months; then $\frac{3}{4}$ of $\frac{5}{6}$ of $\frac{11}{16}=\frac{165}{384}$, and $\frac{11}{61}-\frac{165}{384}=\frac{1584}{6144}=\frac{33}{128}$=£537, therefore, as $\frac{33}{128} : \frac{537}{1} :: \frac{1}{1}$: £2082 18s. $2\frac{2}{11}$d. his fortune.

7. A merchant begins the world with $5000, and finds that by his distillery he clears $5000 in 6 years; by his navigation $5000 in $7\frac{1}{2}$ years, and that in 3 years he spends $5000 in gaming. How long will his estate last him?

$$\text{As}\left\{\begin{array}{l}6\\7\frac{1}{2}\\3\end{array}\right. : 5000 :: 1 : \left.\begin{array}{r}833\frac{1}{3}\\666\frac{2}{3}\\1666\frac{2}{3}\end{array}\right\}\text{Then,}$$

as $1666\frac{2}{3}-833\frac{1}{3}+666\frac{2}{3} : 1 :: 5000 : 30$ years. Ans.

8. A and B are on opposite sides of a circular field 268 rods about; they begin to go round it, both the same way, at the same instant of time; A travels 22 rods in 2 minutes, and B 34 rods in 3 minutes. How many times will they go round the field before the swifter overtakes the slower?

$$\begin{array}{lllll} & \text{min. rods.} & \text{min.} & \text{rods.} & \\ \text{Ans.} & \left.\begin{array}{l}2 : 22\\3 : 34\end{array}\right\} & :: 1 : & \left\{\begin{array}{l}11\\11\frac{1}{3}\end{array}\right. & \begin{array}{l}\text{A goes in a minute.}\\\text{B do. do.}\end{array}\end{array}$$

Therefore, B gains $11\frac{1}{3}-11=\frac{1}{3}$ of a rod of A every minute. And, as $\frac{1}{3}$ rod : 1 min. :: $\frac{268}{2}$ (=half round the field) : 402 min. (=the time in which B will overtake A). Then,

$$\begin{array}{llll} \text{min.} & & & \text{rods.} \\ \text{As } 1 : & \left\{\begin{array}{l}11\\11\frac{1}{3}\end{array}\right. & :: 402 : & \begin{array}{l}\text{4422 A travels.}\\\text{4556 B travels.}\end{array}\end{array}$$

And $\frac{4422}{268}=16\frac{1}{2}$ times round the field, A travels;
and $\frac{4556}{268}=17$ times round the field, B travels.

9. Four persons can perform certain work in the following manner; A, B, and C can do it in 6 days; B

C, and D, in 7 days; A, C, and D, in 8 days; and A, B, and D, in 9 days. In what time can they do it all together; and in what time can each one do it alone? Ans. The power to do the work is inversely as the time; hence, the power of A, B, and C will be $\frac{1}{6}$; of B, C, and D, $\frac{1}{7}$; of A, C, and D, $\frac{1}{8}$; and of A, B, and D, $\frac{1}{9}$. Therefore, $\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}=\frac{1650}{3024}=\frac{275}{504}$ is the power that performs the work three times, for each agent is united to the others three times. Then, $\frac{504}{275}\times3=\frac{1512}{275}=5\frac{137}{275}$ days, the time in which all together will do the work. Therefore, $\frac{275}{1512}-\frac{1}{6}=\frac{138}{9072}$=D's power, by taking A, B, and C's from the sum of the whole power to do the work at once. Then $\frac{9072}{138}$ days,=D's time; in the same way $\frac{9072}{381}$ days is found=to A's time: $\frac{9072}{516}$ days=B's time: and $^{90-2}$=C's time.

10. The hour and minute hands of a watch are exactly together at 12 o'clock. When are they next together? Ans. The velocities of the two hands are to each other as 12 to 1; therefore, the difference of the velocities is 12—1=11.

	o'cl'k.	min.	sec.	
	1	5	$27\frac{3}{11}$	
Hence at	2	10	$54\frac{6}{11}$	&c.
	3	16	$21\frac{9}{11}$	

11. A, B, and C are to share £100 in the proportion of $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{5}$, respectively; but C dying, it is required to divide the whole sum fairly between the two?

£	s.	d.		
57	2	$10\frac{2}{7}$	A's share in all.	Ans.
42	17	$1\frac{5}{7}$	B's share in all.	

12. There are 3 horses employed to draw a load a certain distance, for £2 10s. A and B are supposed to do $\frac{3}{11}$ of the work; A and C $\frac{6}{13}$, and B and C $\frac{4}{13}$ of it; they are paid proportionably. What share does each receive?

Ans. A's share is $19\frac{332}{472}$s.
B's " " $9\frac{102}{472}$s.
C's " " $21\frac{38}{472}$s.

13. A, B, and C have among them 135 guineas; A's + B's are to B's + C's, as 5 to 7: and C's—B's to C's + B's, as 1 to 7. How many had each?

A+B B+C

Suppose A's+B's=50; then, as 5 : 7 :: 50 : 70; and 7 : 1 :: 70 : 10=C's—B's; then 70—10=60, and 60 divided by 2=30=B's; 50—30=20=A's; and 30+10 =40=C's, by the supposition. Now 20+30+40=90, which ought to be 135; therefore,

$$\text{As } 90 : 135 :: \begin{cases} 20 : 30 = \text{A's share.} \\ 30 : 45 = \text{B's do.} \\ 40 : 60 = \text{C's do.} \end{cases}$$

14. A gentleman divided his fortune among his children; to A he gave £8 as often as to B £5; and to C, £3 as often as to B £7; and C's portion was £4000. What was the whole legacy? Ans. £28266 13s. 4d.

15. Sold a watch for £30, and by so doing I lost 15 per cent. whereas I ought to have gained 20 per cent.; what was the watch sold for under its just value?

Ans. £12 $7\frac{3}{51}$s.

16. Two ships sailed from a certain port at the same time; one sailed due north 60 leagues, the other due east 45 leagues. How far were they then asunder?

Ans. 75 leagues.

17. Twenty Generals, 30 Colonels, 24 Majors, and 24 Captains spent at a dinner $320. The expense was divided among them in such a manner, that 4 Generals paid as much as 5 Colonels; 10 Colonels as much as 16 Majors; and 8 Majors as much as 12 Captains. How much money was paid by all the Generals, Colonels, &c.? Ans. The 20 Generals paid $100; the 30 Colonels, $120; the 24 Majors, $60; and the 24 Captains, $40.

18. A can do a piece of work in 6 days; B can do twice as much in 8 days; and C three times as much in 9 days; in what time can they finish it jointly?

Ans. $1\frac{1}{3}$ days.

PERIODICAL DECIMALS.

The reduction of vulgar fractions to decimals presents two cases, which it may be proper to examine. *The first case is where the operation is terminated.* For example, if it be required to reduce $\frac{3}{8}$ to decimals; the operation is as follows,

$$\begin{array}{r|l} 3 & 8 \\ \hline 30 & 0{,}375 \\ 60 & \\ 40 & \end{array}$$

Last remainder 0

Our object here being to obtain a quotient expressed in tenths, hundredths, thousandths, &c., the partial dividends must be converted into the same denominations. As the divisor 8 is not contained in 3, the dividend, we reduce this last to tenths, and have 30 tenths to be divided by 8, which gives 3 tenths for quotient, and 6 tenths, or 60 hundredths, for remainder; dividing 60 hundredths by 8, we have 7 hundredths for quotient, and 4 hundredths, or 40 thousandths, for remainder; then dividing 40 thousandths by 8, we have 5 thousandths for quotient, and cipher for the last remainder. The value of $\frac{3}{8}$ in decimals, then, is 0,375 or 375 thousandths. This result may be verified by reducing the fraction $\frac{375}{1000}$ to its most simple expression, for $\frac{3}{8}$ will then be reproduced.

The second case is where the operation is not terminated. For example, if the fraction $\frac{3}{11}$ be reduced to decimals, the operation and result are as follows.

$$\begin{array}{r|l} 3 & 11 \\ \hline 30 & 0{,}272727, \&c. \\ 80 & \\ 30 & \\ 80 & \\ 30, \&c. & \end{array}$$

As in the preceding case, the alternate remainders 3 and 8, express, successively, tenths, hundredths, thousandths, ten thousandths, hundred thousandths, &c.— These remainders diminish in value very rapidly; hence, by adding a sufficient number of terms to the quotient 0,272727, &c., we can obtain the value of the fraction $\frac{3}{11}$ approximated to any required degree of exactness. This is a general remark. For by precisely the same operation we find, that the value of $\frac{1}{7}$, expressed in decimals, is 0,142857 142857, &c., and that of $\frac{12}{37}$, 0,324 324, &c.

Decimal fractions of this kind, in which a certain number of figures are repeated periodically in the same order, without terminating, are called, from this property, periodical fractions.

As a property of this kind of numbers, it may be observed, that *the number of different remainders, obtained by division, can never exceed the number of units, contained in the divisor.* For, by the nature of the operation, the remainders, being necessarily less than the divisor, will be found in the series of natural numbers between cipher and that which precedes the divisor. Thus, for example, the divisor being 7, we can only obtain these 7 different remainders, 0, 1, 2, 3, 4, 5, 6. Hence, after a number of divisions equal to, or less than, the number of units in the divisor, there will result a remainder, which is cipher, or a remainder already obtained. In the former case, the fraction will be terminated; but in the latter, the figures of the quotient returning in the same order, without terminating, the fraction will be periodical.

To know whether a vulgar fraction can be expressed exactly in decimals, or not,—it may be remarked, that *every fraction, the denominator of which is not divisible either by 2 or by 5, cannot be expressed exactly in decimals, and, consequently, gives rise to a periodical fraction.* As this is an important property, we shall give two general demonstrations of it.

1st. In performing the division of the numerator by the denominator, in order that the operation may terminate, one of the remainders, (none of which are divisible by the divisor,) must be made so; it is necessary, then, that the multiplication of these remainders by 10, should introduce some factor common with those in the divisor; which can take place only when the divisor contains one of the factors 2 and 5, that compose the number 10.

2d. In performing the division of the numerator by the denominator, that the operation may terminate, some one of the partial dividends must be divisible by the divisor, and, consequently, must contain all its factors in the same number. But, the successive remainders being smaller than the divisor, can never contain the same factors in the same number; it is necessary, then, that the multiplication of these successive remainders should introduce the factors, that are wanting; that is, the factors of the dividend, which are not found in the remainder. But, 10 being divisible only by 2 and 5, cannot introduce these factors. Then, if no one of them is contained in the divisor, on multiplying the remainders by 10 to form the partial dividends, we do not introduce any factors, which are common with those in the divisor; consequently the division will not be exact, and the quotient will be periodical.

When a periodical fraction is given, if we neglect the whole series after the first period, we shall have a second decimal fraction composed of the first period, and, consequently, equal to a common fraction having this period, considered as a whole number, for its numerator, and for a denominator, unity, followed by as many ciphers as there are figures in the period. Hence, the numerator of this last fraction remaining always equal to the period, the denominator is diminished by unity. And as this denominator is equal to unity followed by as many ciphers as there are figures in the period, it will then become a number composed of as many nines as there are figures in the period. The new fraction which results will then have the period for numerator, and for denominator, as many nines as there are figures in the period. This last fraction, having the same numerator as the preceding, and a denominator less by unity, on performing division, will give a quotient, that is the same as the preceding, that is, the first period; and the remainder, instead of being a cipher, will be equal to the quotient, or that same period. The remainder being the same as the first dividend, the operation will be recommenced in the same manner, since it will be reduced to a continual multiplication by 10. The figures already obtained in the quotient (the same that compose the period) will then be continually reproduced in the same order, and, consequently will form a periodical fraction equal to the given one, for it will be composed of the same period. But this periodical fraction is derived from a common fraction, having the period for numerator, and for denominator, a number composed of as many nines as there are figures in the period. Hence, we deduce this general rule. *Every periodical fraction is equal to a common fraction, having the period for numerator, and for denominator, a number composed of as many nines as there are figures in the period.* If we apply this rule to the following periodical fractions,

0,142857 142857, &c. and 0,324 324, &c.

for their respective values, we shall have the fractions $\frac{142857}{999999}$, and $\frac{324}{999}$.

These fractions, reduced to their most simple expressions, are $\frac{1}{7}$ and $\frac{12}{37}$.

In the preceding operations, we have supposed the period to commence with the first figure; when it does not commence with the first figure, the following method is to be followed. Let 0,48 324 324 be a periodical fraction, of which the period 324 does not commence until after the second decimal. We multiply this fraction by 100, by removing the separatrix two places towards the right, which gives 48,324 324; or 48 plus 0,324 324, &c.; but, the periodical fraction 0,324 324, &c., of which the period commences with the first figure, has for its value the fraction $\frac{324}{999}$. Then the periodical fraction 48,324 324 has for its value 48 whole numbers and $\frac{324}{999}$ or $\frac{48276}{999}$, and as this last fraction is 100 times greater than the given one, the given fraction will be equal to $\frac{48276}{99900}$.

If we generalize the reasoning, that has conducted to this result, the following rule may be deduced.

When the period does not commence until after several decimals, the separatrix must be removed towards the right, unto the first figure of the period, which is the same as to multiply the given periodical fraction by unity, followed by as many ciphers as there are figures before the period. We then have a new decimal fraction, composed of a whole number, plus a periodical fraction, whose period commences with the first figure. This last fraction, joined to the whole number, gives a fractional expression, which is equal to the given periodical fraction, multiplied by unity, followed by as many ciphers as there are figures before the period. Then dividing this expression by unity, followed by as many ciphers as there are figures before the period, the result will be the value of the given periodical fraction.

LOGARITHMS.

216. Logarithms are a series of numbers in arithmetical progression, corresponding, term for term, to a similar series in geometrical progression. For example, in the two following series :—

∺ 2 : 4 : 8 : 16 : 32 : 64 : 128 : 256 : &c.

∺ 3 · 5 · 7 · 9 · 11 · 13 · 15 · 17 · &c.

Each term in the lower, or arithmetical series, is called the logarithm of the term corresponding to it, in the upper, or geometrical series.

217. Any number may have a great variety of different logarithms ; because, to any geometrical progression, there may be formed a great number of corresponding, different, arithmetical progressions. But as our design is to consider logarithms only in relation to their use in numerical calculations, we shall not consider the different arithmetical and geometrical progressions, that might be compared together ; but pass to the consideration of such as are used in the formation of Tables of Logarithms.

218. For the geometrical progression, the decuple or tenfold progression is chosen ; and for the arithmetical

progression, the series of natural numbers; that is, the two following progressions are selected.

∺ 1 : 10 : 100 : 1000 : 10000 : 100000 : 1000000.

∺ 0 · 1 · 2 · 3 · 4 · 5 · 6

219. The logarithm of an unit followed by any given number of ciphers, may always be easily determined; for it will consist of as many units as there are ciphers following that unit.

We shall not explain the method for finding the logarithms of the intermediate terms in the decuple progression. It depends upon principles, with which the learner is not yet supposed to be acquainted, But we will explain the formation of logarithms by a method, which is not the most expeditious for calculating them, but suffices, as well to give an idea of their formation, as to afford a reason for the uses, to which these artificial numbers are applied.

220. After the definition which has been given of logarithms, it is plain, that, to have the logarithm of any number, (of 3, for example,) this number must form a part of the fundamental geometrical progression. And although 3 does not appear to form a part of this geometrical progression ∺ 1 : 10 : 100 : 1000 : &c.; yet it may be seen in article (214,) that between 1 and 10 a great number of geometrical means may be inserted. Then, as we ascend from 1 to 10 by gradations the more closely united, that is, by numbers, that differ from each other the less, in proportion as the number of such means shall be greater; either some one of these means will be precisely the number 3, or two consecutive numbers will be found, between which the number 3 is contained, and of which each shall differ as little from 3, as the number of means inserted shall be greater. Then, in like manner, if we insert between 0 and 1 as many arithmetical means as there were geometrical means inserted between 1 and 10, each term of the geometrical progression will have for its logarithm, the term, which corresponds to it in the arithmetical progression. That is, for the logarithm of 3, we take the number, that is found in the same place, in which 3 is found in the geometrical progression.

But if no one of these numbers be exactly the number 3, we take, in the arithmetical progression, the term, that corresponds to that term of the geometrical progression, which approaches nearest to the number 3.

On this principle is founded the calculation of logarithms; and although more expeditious methods may be followed, the results will be the same.

221. Insert, or suppose to be inserted, 10000000 geometrical means between 1 and 10, the same between 10 and 100, the same number between 100 and 1000, &c.; and then insert the same number of arithmetical means between 0 and 1, the same number between 1 and 2, between 2 and 3, &c.; then, having placed all the first in one line, and all the second underneath them, seek among the first, the number 2, or number nearest to it, and take the corresponding number in the lower series. In like manner, seek among the first series the number 3, or number nearest to it, and take the corresponding number in the lower series. Do the same successively, for the numbers 4, 5, 6, &c. and writing the numbers 1, 2, 3, 4, 5, &c. in the same vertical column, (as may be seen in the annexed Table,) place in a column by the side of them, the terms of the arithmetical progression, which have been found to correspond to these numbers, or to the numbers, which approach nearest to them. From this operation an idea may be obtained of the formation of Logarithms, and of their arrangement in common Tables.

222. It may be remarked concerning logarithmic tables, that the first figure to the left of each logarithm is called the *characteristic;* because it is by this figure, that we know in what decade the number is contained, to which the logarithm belongs. For example, if a number have for its characteristic, the number 3, we know this number must be a certain number of thousands; because, the logarithm of 1000 being 3, and that of 10000 being 4, every number between 1000 and 10000 must have for its logarithm, 3 and a fraction. It will then have 3 for its characteristic; and the other figures express the fraction reduced to decimals.

PROPERTIES OF LOGARITHMS.

223. As we treat of only such logarithms as are found in ordinary Tables, our explanations will be confined to such properties as belong to geometrical progressions, that have unity for their first term, and arithmetical progressions, that have cipher for their first term.

Let us compare term for term, any geometrical progression whose first term is unity, with any arithmetical progression whose first term is cipher. For example, the two following progressions.

∺ 1 : 3 : 9 : 27 : 81 : 243 : 729 : 2187 : 6561 : &c.

÷ 0 · 4 · 8 · 12 · 16 · 20 · 24 · 28 · 32 · &c.

It is inferred from the nature and perfect correspondence of these two progressions, that as often as the ratio of the first progression is factor in any one of its terms, so often will the ratio of the second progression be contained in the term corresponding to the term considered in the first progression. For example, in the term 2187, the ratio 3 is seven times factor; and in 28 the term corresponding to it, the ratio 4 is contained seven times. And, according to what has been shown in articles (206 and 212), the ratio is factor in any term of the first progression, as often as there are terms preceding it; and in the second progression, any term is composed of as many times the ratio, as there are terms preceding it. But the number of terms is the same in both progressions. Hence, any term of a geometrical progression will always have for its corresponding term in the arithmetical progression, a term which shall contain the ratio of this progression as often as the ratio of the other progression is factor in the first.

224. Then, *if we multiply together any two terms of a geometrical progression, and at the same time add together the two corresponding terms of the arithmetical progression, the product and sum will be two corresponding terms in these progressions.*

For it is plain, that the ratio will be factor in the product, whatever it is, as often as in both the terms that are multiplied together; and the ratio of the arithmetical progression will be contained in the sum, whatever it is, as often as in both the terms that are added together.

225. Then, by the addition only of two terms of an arithmetical progression, we may know the product of the two corresponding terms of the geometrical progression; by supposing the progressions to be sufficiently extended.

For example, by adding the terms 8 and 24, which correspond to 9 and 729, we have 32 for the sum, which corresponds to 6561. Hence we infer, that the product of 729 by 9 is 6561; which is the case.

Then as the natural numbers that compose the first column of the annexed Table, have been taken from a geometrical progression, which commences with unity; and as their logarithms are the corresponding terms of an arithmetical progression that commences with cipher; we may infer, that *by adding together the logarithms of two numbers, we obtain the logarithm of their product.* Hence we deduce the following uses.

USES OF LOGARITHMS.

227. *To perform multiplication by logarithms, add the logarithm of the multiplicand to the logarithm of the multiplier, and their sum will be the logarithm of the product; seek for this sum among the logarithms of the Table, and by the side of it will be found the product.* For example, if it be required to multiply 14 by 13, we seek in the table for the logarithm of 14, which is
1,146128
and that of 13, which is 1,113943

The sum 2,260071 corresponds in the same Table to the number 182, which is the product of 14 by 13.

228. Then to square a number, we need only double its logarithm; for, to multiply any number by itself, the logarithm of that number is added to itself.

229. For the same reason, to cube a number, we triple its logarithm; and in general, to raise a number to any power whatever, we take the logarithm of that number as

many times as there are units in the number, that denotes the power. That is, we multiply its logarithm by the number, that expresses the power. For example, to raise a number to the seventh power, we multiply its logarithm by the number 7.

230. Then reciprocally, to extract the square, cube, fourth, &c. root of a given number, we must divide that number by 2, 3, 4, &c.; that is, in general, by the number that expresses the degree of the root to be extracted. For example, if the square root of 144 be required, we find by the Table, that the logarithm of this number is 2,158362; then taking 1,079181 the half of this logarithm, we find that 12 is the number corresponding to it; which is the square root of 144.

If the seventh root of 128 be required, we find by the Table, the logarithm of this number is 2,107210; then taking the seventh part of it, we seek in the Table what number corresponds to 0,301030, the quotient, and find it is 2; which is the seventh root of 128.

231. To find the quotient of the division of one number by another, *subtract the logarithm of the divisor from the logarithm of the dividend; the number that corresponds to the logarithm of the remainder, will be the quotient.* For example, if it be required to divide 187 by 17, we seek in the Table for the logarithms of these two numbers, and find for the logarithm of 187,

2,271842
and for that of 17 1,230449

The difference 1,041393 corresponds in the Table to 11; which is the quotient of 187 divided by 17.

When division cannot be performed without a remainder, the whole logarithm of the remainder is not used. We shall show hereafter what must be done in such cases.

The reason for this rule is founded on the principle established in article (74), that the quotient multiplied by the divisor, produces the dividend; then, (227) the logarithm of the quotient, added to the logarithm of the divisor will produce the logarithm of the dividend; consequently, the logarithm of the quotient is equal to the logarithm of the dividend, minus the logarithm of the divisor.

232. After what has been explained, it is evident, that, to perform the Rule of Three by logarithms, we must add the logarithm of the second term to the logarithm of the third, and subtract from their sum the logarithm of the first term.

233. We may remark, that when the logarithm, that results from operations upon other logarithms, is sought in the Table; if this logarithm differ from that in the Table only in the last figure, this difference is to be considered as nothing; for all intermediate numbers in the decuple progression, are approximated only to half an unit in the seventh place of decimals.

NUMBERS WHOSE LOGARITHMS ARE NOT FOUND IN THE TABLES.

234. The logarithms of fractions, and of whole numbers joined to fractions, are not found in the Table; the same is true of the square roots, cube roots, &c. and also of numbers, which are not perfect powers to the degree of these roots.

When the logarithm is required of a whole number joined to a fraction, it must be reduced to an improper fraction (86), and then the logarithm of the denominator subtracted from the logarithm of the new numerator. For example, to obtain the logarithm of $8\frac{3}{11}$, we seek that of $\frac{91}{11}$, and find for 11, the logarithm 1,041393; and for 91, the logarithm 1,959041; then 0,917648, the difference between those logarithms, will be the logarithm of $8\frac{3}{11}$; for $8\frac{3}{11}$, or $\frac{91}{11}$, is the same as 91 divided by 11 (96).

235. For the same reason, to obtain the logarithm of a fraction, we must, in like manner, subtract the logarithm of the denominator from the logarithm of the numerator; but as this subtraction cannot be performed when the logarithm of the denominator is largest, we then subtract the logarithm of the numerator from that of the denominator; the remainder, (which is the quantity, that was wanting in the first case to allow subtraction to be made,) will be the logarithm of the fraction, by applying to it

the minus sign. Thus, the logarithm of the fraction $\frac{11}{91}$ is—0,917648. Numbers preceded by this sign are called *negative* numbers.

236. The use of this sign is to show, that the logarithms of the fractions to which it is applied, must be employed according to a rule, contrary to that given for the logarithms of whole numbers; or of whole numbers joined to fractions. That is, if we are to multiply by a fraction, the logarithm of the fraction must be subtracted; on the contrary, if we are to divide by a fraction, its logarithm must be added. For multiplication, the reason of this change is, that to multiply by a fraction, is the same thing as to multiply by the numerator and then divide by the denominator; then, when the operation is by logarithms, we must add the logarithm of the numerator, and subtract that of the denominator; or, what is the same thing, we subtract the excess of the logarithm of the denominator above that of the numerator; but, this excess is the logarithm of the fraction. For division, the reason of the operation is also plain; for, to divide by $\frac{3}{4}$, for example, is the same thing (109) as to multiply by $\frac{4}{3}$; then, when operating by logarithms, we must add the logarithm of $\frac{4}{3}$; that is, (134) the difference between the logarithms of 4 and 3, or the difference between the logarithms of the denominator and numerator of the given fraction.

237. It often happens, when a whole number and fraction, whose logarithm is sought, is changed into an improper fraction, that the numerator is a number, that exceeds the limits of the Table. For example, if the logarithm of $53\frac{821}{5704}$ be required; this number when reduced to a fraction becomes $\frac{303133}{5704}$, the numerator of which exceeds the limits of ordinary Tables. We shall explain a method for finding the logarithms of such numbers, and although it may not be strictly rigorous, it is sufficiently so for common uses. It may, however, be observed:

238. 1st. That by adding 1, 2, 3, &c. units to the characteristic of the logarithm of a number, we multiply that number by 10, 100, 1000, &c.; because it is to add to it the logarithm of 10, 100, 1000, &c. (219 and 227).

2d. On the contrary, if 1, 2, 3, &c. units be subtrac ed from the characteristic of a logarithm, it is the same as to divide the number corresponding to that logarithm, by 10, 100, 1000, &c.

239. Now if it be required to find the logarithm of 357859, for example; we separate by a comma upon the right of this number, as many figures as are necessary to bring it within the limits of the Table. In this case we separate two, which gives 3578,59; this number (28) is 100 times smaller than 357859. We find in the Table 3,5536403* for the logarithm of 3578; and then take the difference 1214 between this logarithm and that of 3579. We then make the following proportion in the Rule of Three: if for 1 unit, the difference between the numbers 3579 and 3578, we have 1214 for the difference of their logarithms; how great, for 0,59, the difference between the numbers 3578,59 and 3578, will be the difference of their logarithms? That is, we seek the fourth term of a proportion, whose three first terms are,

$$1 : 1214 :: 0{,}59 :$$

The fourth term of this proportion is 716,26, or simply 716, by neglecting the decimals. We then add 716 to 3,5536403, the logarithm of 3578, and have 3,5537119 for the logarithm of 3578,59. Now for the logarithm of 357859, we have only to add two units to the characteristic of the logarithm already found; and have 5,5537119 for the logarithm sought; for 357859 is then made 100 times greater than 3578,59.

If all the figures separated on the right had been ciphers; after having found in the Table the logarithm of the remaining part, it would have been sufficient to have added as many units to the characteristic, as there were ciphers separated.

240. If the logarithm be required of a number accompanied by decimals, we seek for the logarithm of the whole number, as if no part of it were decimals; and after having found it, either immediately by the Table, or by the method explained in article (239), we take as many units from the characteristic as there are decimals in the given number. For, having considered the number as if no part were decimals; that is, as being 10, 100, 1000, &c. times greater than it really is; we reduce it

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[*] These and many of the following numbers and logarithms are not found in the annexed table.

again to its true value, by diminishing the characteristic of its logarithm (238).

241. When only decimals are in the given number, we seek for the logarithm belonging to it, as if it were a whole number; and after subtracting from it as many units as there are decimals in the given number, prefix the sign minus to the remainder. For example, to obtain the logarithm of 0,03, we seek for that of 3, which is 0,477121; from this we subtract two units, and applying the sign to the remainder, have —1,522879 for the logarithm of 0,03. The true value of 0,03 is $\frac{3}{100}$; but, to obtain the logarithm of $\frac{3}{100}$, we must (235) subtract the logarithm of 3 from that of 100, and affix the sign minus to the remainder.

LOGARITHMS WHOSE NUMBERS ARE NOT FOUND IN THE TABLES.

242. The following investigation is not less necessary than the preceding. For example; it seldom happens that the quotient of division is a whole number; but when the operation is performed by logarithms, the logarithm of the quotient cannot be found in the Table, except it be a whole number. Other similar cases frequently occur.

243. We first propose to find, to what number a given logarithm corresponds whether it exceed the limits of the Table, or fall among the logarithms contained in it. For this purpose, we subtract from the characteristic of the logarithm, as many units as are requisite, in order that the given logarithm thus prepared may be found in the Table. Then, if all the figures of the logarithm be found in the Table, the number sought will be the same, that is found in the Table corresponding to the logarithm, after adding as many ciphers to it as there were units taken from the characteristic (238). For example, the logarithm 7,2273467, after three units are taken from its characteristic, corresponds to the number 16879; hence we infer, that the given logarithm 7,2273467 corresponds to 16879000.

If only the first figures of the logarithm are found in the Table, the operation is performed as in the following example. To find to what number the logarithm 5,2432768 belongs, we take two units from its characteristic; the logarithm 3,2432768, which we then have, falls between the logarithms of 1750 and 1751; the number to which it corresponds is then 1750 and a fraction. To find this fraction, we subtract from the logarithm 3,2432768, the logarithm of 1750, and have 2388 for the difference. We also take from the Table, the difference 2481, between the logarithms of 1751 and 1750, and then make this proportion in the Rule of Three. If 2481, the difference between the logarithms of 1751 and 1750, corresponds to 1 unit, the difference between those numbers; to what difference of numbers, will 2388, the difference between the logarithm 3,2432768 and that of 1750, correspond? We find $\frac{2388}{2481}$ for the fourth term of the proportion. Thus the logarithm 3,2432768 belongs to $1750\frac{2388}{2481}$ very nearly. Consequently, the given logarithm, which belongs to a number 100 times greater (238), has for its corresponding number $175000\frac{238800}{2481}$; that is, $175096\frac{624}{2481}$; or, by reducing it to decimals, 175096,25.

244. When the given logarithm falls within the Table, the characteristic need not be diminished; consequently, there will be no ciphers to add at the end of the operation; which is performed in the usual manner.

245. But as the proportion, which has been used in this method, is not rigorously exact (for it supposes, that the differences of the logarithms are proportional to the differences of their numbers, which is not strictly true, but sufficiently so for ordinary uses); and as it does not give the requisite approximation, only when the numbers employed are considerably large; for greater exactness, it will be necessary, when the given number is below 1500, to add as many units to its characteristic as shall carry it beyond the limits of the Table. Then, having found the nearest number that corresponds to it in the Table, we separate as many figures upon the right of that number as there were units added to the characteristic. In general, this method is sufficient. But when a greater number of decimals is required, we make a proportion, as directed in article (243,) and reducing the fourth term to decimals, place it after those already

found. For example; if it be required to know what number corresponds to the logarithm 0,5432725; as this logarithm falls between the logarithms of 3 and 4, and as the number to which it belongs is, consequently, much below 1500, we seek for this logarithm with three units added to its characteristic, that is, we seek for 3,5432725. This logarithm falls between the logarithms of 3493 and 3494, hence we infer, that the number sought is 3,493; to three places of decimals. But, if this approximation is not sufficient, we take 739, the difference between 3,5432725 and the logarithm of 3493; and also 1243, the difference between the logarithms of 3494 and 3493, and seek, by reasoning as in article (243,) the fourth term of a proportion, that commences with these three,

$$1243 : 1 :: 739 :$$

The fourth term, in decimals, is 0,594; then the number sought is 3,493594. Finally, this second approximation is limited; for, the logarithms of the Table being exact only to half a decimal unit of the seventh order, the differences are affected by this slight defect. The approximation, however, may be extended with confidence to the third place of decimals, and we seldom have occasion to continue it further. This remark is also applicable to the use, that was made of the same proportion in articles (239 and 243).

246. If the fraction be required, to which a given negative logarithm corresponds, we subtract from the logarithm 1, 2, 3, 4, &c. units, according to the extent of the Table; and after having found the number, that corresponds to the remaining logarithm, we separate upon the right of it, as many figures as there were units in the number whose logarithm was subtracted. For example, if it be required to know what fraction belongs to this logarithm — 1,532732, we subtract 1,532732 from 4, and there remains 2,467268, which in the Table is found between the logarithms of 293 and 294; hence we infer, that the fraction sought is between 0,0293 and 0,0294; that is, it is 0,0293, to four places of decimals. But, to subtract the given logarithm 1,532732 from 4, is (236) to multiply 10000 by the fraction to which this logarithm belongs; or, it is the same thing as to multiply this

fraction by 10000. Then, the number which has been found, is 10000 times too great; it must, therefore, be considered as composed of ten thousandths of an unit, and is expressed by four decimal figures.

We now proceed to show, by some examples, the advantages that logarithms afford for facility in calculations.

EXAMPLE I.

The quotient of 17954 divided by 12836 is required to four places of decimals.

The logarithm of 17954 is	4,254161
The logarithm of 12836 is	4,108430
Remainder	0,145731

This remainder, with a characteristic augmented by four units, corresponds in the Table to 13987; then (238) the quotient sought is 1,3987.

EXAMPLE II.

The cube root of 53 is required to three places of decimals.

The logarithm of 53 is	1,724276
Its third part (230) is	0,574759

This last logarithm sought in the Table with a characteristic augmented by three units, corresponds to 3756; then (238) the root sought is 3,756.

To judge of the important use of logarithms, we need only seek for this root by the method given in article (156). That method, however, ought not to be considered useless; because it applies to an infinity of numbers, to which logarithms do not, by reason of the limits of the Tables.

EXAMPLE III.

The fifth root of the cube of 5736 is required to two places of decimals.

We triple 3,758609, the logarithm of 5736, and have 11,275827, for the logarithm of the cube of 5736. Taking the fifth of this last logarithm, we have 2,255165 for the logarithm of the fifth root of the cube of 5736. This logarithm, sought in the Table with a characteristic augmented by two units, corresponds to a number between 17995 and 17996; the root required is then 179,96, to two places of decimals.

EXAMPLE IV.

It is required to find four geometrical mean proportionals between $2\frac{2}{3}$ and $5\frac{3}{4}$.

According to the method given in article (215,) the ratio between the terms of this progression must be found by dividing $5\frac{3}{4}$ by $2\frac{2}{3}$, and extracting the fifth root of the quotient. But by logarithms this operation is very simple. We find by the tables, that the logarithm of $5\frac{3}{4}$ or $\frac{23}{4}$ is 0,759668. In like manner, we find, that the logarithm of $2\frac{2}{3}$ is 0,425969. We then subtract (231) this logarithm from the first, and have 0,333699. Then taking the fifth of this last, have 0,066740 for the logarithm of the ratio sought. This logarithm, found in the tables, with a characteristic augmented by 4 units, for 4 decimals, corresponds to 11661 in whole numbers. The ratio, then, is 1,1661, to four places of decimals. Now, to obtain the required mean proportionals, we multiply $2\frac{2}{3}$, the first term, by 1,1661; then, the product of that multiplication by 1,1661; and so on. But these operations may be more readily performed by the aid of logarithms, by adding to 0,425969, the logarithm of $2\frac{2}{3}$, the first term, the logarithm 0,066740 of the ratio; then its double, triple, and quadruple; hence we have 0,492709; 0,559449; 0,626189; 0,692929; for the logarithms of the four required geometrical mean proportionals. And if we seek for these logarithms in the Tables; with three units for their characteristic, we shall find, that these four mean proportionals are 3,109; 3,626; 4,288; 4,931.

REMARK.

In an operation performed by logarithms, where subtraction is required, the operation may be simplified by the following method.

When any number is to be subtracted from another, which is an unit followed by as many ciphers as there are figures in the first; the operation may be performed by taking the difference between 9 and each of the figures of the given number, except the last; for which, we write the difference between 10 and that figure. For example, if 526927 is to be subtracted from 1000000; we subtract successively the figures 5, 2, 6, 9, 2, from 9, and the last figure, from 10, and have 473073, for a remainder. This remainder is called the *arithmetical complement* of the given number.

As this method of subtraction is too simple to be called an operation, it follows, when a result is to be obtained from the addition and subtraction of many numbers, that the operation may be reduced to addition. For example, if it be required to add these two numbers 672736, 426452, and subtract from their sum these other two numbers 432752, 18675; instead of one addition and two subtractions, which would be required by the common method, we substitute the following,

	672736
	426452
Arithmetical complement of 432752	567248
Arithmetical complement of 18675	981325
Sum	2647761

That is, we add together the two first given numbers, and the arithmetical complements of the two last, and their sum is 2647761. The first figure 2 must be suppressed; and the remaining figures 647761 is the result sought.

The reason of this operation is plain; for if, instead of subtracting 432752, we add its arithmetical complement, that is, 1000000 minus 432752, we at the same time perform the required subtraction, and augment the sum by 1000000; that is, by one tenth too much in the first figure of the result. Then, for each arithmetical complement used in the operation, we shall have one tenth too much in the first figure of the result.

We will now show the application of this principle to logarithms. For example, if it be required to divide 3760 by 79; by the ordinary method, the logarithm of 79 would be subtracted from that of 3760. But instead

of that operation we, substitute the following, and write
the logarithm of 3760 3,575188
and the arith. complt. of the log. of 79 8,102373

Sum 11,677561

Thus, 1,677561 is the logarithm of the quotient, and corresponds to 47, 59, to two places of decimals.

For a second example, let it be required to multiply $\frac{675}{527}$ by $\frac{952}{377}$. According to the usual method (106), 675 must be multiplied by 952; and 527 by 377, then the first product divided by the second.

By logarithms, the operation is as follows.

Logarithm of 675	2,829304
Logarithm of 952	2,978637
Arith. complt. of the log. of 527	7,278189
Arith. complt. of the log. of 377	7,423659
Sum	20,509789

The logarithm of the product is 0,509789, which with three units for a characteristic, corresponds to 3,234.

We can also use the arithmetical complement, in order to give to the logarithms of fractions the same form as those of whole numbers, and employing them in the same manner, avoid, in making calculations, the distinction of positive and negative logarithms. It may be well to recollect, that the characteristic of the logarithms of fractions, properly so called, is too great by 10 units. For example, to obtain the logarithm of $\frac{3}{4}$, which is no other (96) than 3 divided 4; instead of subtracting the logarithm of 4 from that of 3, that is, the logarithm of 3 from that of 4, and affixing the sign minus to the difference (235); we add to the logarithm of 3, the arithmetical complement of the logarithm of 4; thus,

The logarithm of 3 is	0,477121
Arith. complt. of the log of 4	9,397940
Sum	9,875061

This sum is the logarithm of $\frac{3}{4}$, the characteristic of which is too large by 10 units. But we need not actually diminish it by 10 units; for so many may be rejected at the end of the operations, in which the logarithm was used.

The same rule applies also to decimal fractions. Thus to obtain the logarithm of 0,575, which is no other than $\frac{575}{1000}$, we add to the logarithm of 575, the arithmetical complement of the logarithm of 1000. And by employing the arithmetical complements, instead of the negative logarithms of fractions, the values of the fractions, expressed in decimals, may be found without difficulty in the Table. And when we know, that the given logarithm is, or contains, one or more arithmetical complements, we know that its characteristic is too large, by as many tens as there are arithmetical complements in the logarithm. Thus, if it exceed such a number of tens, it will be easy to diminish it, and find the number, that belongs to the logarithm; which will be a whole number, or a whole number joined to a fraction. But if the characteristic be below the number of tens, of which it is supposed to contain too many, it belongs to a fraction, which may be found in the following manner. We seek, according to what was explained in article (242 and following), to what number the given logarithm corresponds; and when found, we separate upon the right of it, as many tens of figures, as is the excess of tens in the characteristic. For example, if we have 8,732235, for the logarithm resulting from an operation, in which there was an arithmetical complement; we perceive, as its characteristic is less than one ten, that it belongs to a fraction. We first seek (243) to what number 8,732235, considered as the logarithm of a whole number, corresponds; and find, that it corresponds to 539802600; separating 10 figures from this number, we have 0,0539802600, for the approximate value of the fraction, to which the given logarithm corresponds.

But, as it is very seldom necessary to obtain the fractions to such a degree of precision as in the last example, we abridge the operation by diminishing the characteristic of the given logarithm, as many units as will bring it within the Table; and then taking the corresponding number, we separate from it *as many figures less* than was required by the preceding rule, as units were taken from the characteristic. Thus, using the last example, we diminish the characteristic 5 units, and finding that the corresponding logarithm is 5398, we separate 5 figures only, and have 0,05398.

In the elevation of powers, we must observe, that by multiplying (229) the logarithm by a number, which expresses the degree of the power, we shall multiply a logarithm whose characteristic is too large. Thus, for example, in raising a number to its cube, if there be an arithmetical complement in the given logarithm; that is, if the characteristic be too great by 10 units, the characteristic of the logarithm of the cube will be too great by 30 units; and so of the other powers. It will then be easy to reduce it to its just value.

In the extraction of roots, to prevent mistake when there are arithmetical complements in the logarithms used, care must be taken to add to, or subtract from, the characteristic, such a number of tens as will make it too large by as many tens as there are units in the number, that expresses the degree of the root. And having, according to the common rule, divided by the number, that expresses the degree of the root, the characteristic will be too great by precisely 10 units. For example, if the cube root be required of $\frac{276}{547}$; we add to the logarithm of 276, the arithmetical complement of the logarithm of 547.

Logarithm of 276	2,440909
Arith. complt. of the log of 547	7,262013
Sum	9,702922

To the characteristic of this logarithm we add 20; it then becomes 29,702922, and is too great by 3 tens. The third part of this logarithm, which is, 9,900974, is the logarithm of the cube root required, but with a characteristic too large by 10 units. Thus, according to what has been said above, we find that the cube root is 0,7961, to four places of decimals.

Arithmetical complements are chiefly used in Trigonometrical calculations.

TABLE OF LOGARITHMS

Of the natural numbers from 1 to 200.

Numbers.	Logarithms.	Numbers.	Logarithms.	Numbers.	Logarithms.	Numbers.	Logarithms.
0	Inf. neg.	39	1,591065	78	1,892095	117	2,068186
1	0,000000	40	1,602060	79	1,897627	118	2,071882
2	0,301030	41	1,612784	80	1,903090	119	2,075547
3	0,477121	42	1,623249	81	1,908485	120	2,079181
4	0,602060	43	1,633468	82	1,913814	121	2,082785
5	0,698970	44	1,643453	83	1,919078	122	2,086360
6	0,778151	45	1,653213	84	1,924279	123	2,089905
7	0,845098	46	1,662758	85	1,929419	124	2,093422
8	0,903090	47	1.672098	86	1,934498	125	2.096910
9	0,954243	48	1,681241	87	1,939519	126	2,100371
10	1,000000	49	1,690196	88	1,944483	127	2,103804
11	1,041393	50	1,698970	89	1,949390	128	2,107210
12	1,079181	51	1,707570	90	1,954243	129	2,110590
13	1,113943	52	1,716003	91	1,959041	130	2,113943
14	1,146128	53	1,724276	92	1,963788	131	2,117271
15	1,176091	54	1,732394	93	1,968483	132	2,120574
16	1,204120	55	1,740363	94	1,973128	133	2,123852
17	1,230449	56	1.748188	95	1,977724	134	2,127105
18	1,255273	57	1,755875	96	1,982271	135	2,130334
19	1,278754	58	1,763428	97	1,986772	136	2,133539
20	1,301030	59	1,770852	98	1,991226	137	2,136721
21	1,322219	60	1,778151	99	1,995635	138	2,139879
22	1,342423	61	1,785330	100	2,000000	139	2,143015
23	1,361728	62	1,792392	101	2,004321	140	2.146128
24	1,380211	63	1,799341	102	2,008600	141	2,149219
25	1,397940	64	1,806180	103	2,012837	142	2,152288
26	1,414973	65	1,812913	104	2,017033	143	2,155336
27	1,431364	66	1,819544	105	2,021189	144	2,158362
28	1,447158	67	1,826075	106	2,025306	145	2,161368
29	1,462398	68	1,832509	107	2,029384	146	2,164353
30	1,477121	69	1,838849	108	2,033424	147	2.167317
31	1,491362	70	1,845098	109	2,037426	148	2,170262
32	1,505150	71	1,851258	110	2,041393	149	2,173186
33	1,518514	72	1.857332	111	2,045323	150	2,176091
34	1,531479	73	1,863323	112	2,049218	151	2,178977
35	1,544068	74	1,869232	113	2,053078	152	2,181844
36	1,556303	75	1,875061	114	2,056905	153	2,184691
37	1,568202	76	1,880814	115	2,060698	154	2,187521
38	1,579784	77	1,886491	116	2,064458	155	2,190332

Numbers.	*Logarithms.*	*Numbers.*	*Logarithms.*	*Numbers.*	*Logarithms.*	*Numbers.*	*Logarithms.*
156	2,193125	167	2,222716	178	2,250420	189	2,276462
157	2,195900	168	2,225309	179	2,252853	190	2,278754
158	2,198657	169	2,227887	180	2,255273	191	2,281033
159	2,201397	170	2,230449	181	2,257679	192	2,283301
160	2,204120	171	2,232996	182	2,260071	193	2,285557
161	2,206826	172	2,235528	183	2,262451	194	2,287802
162	2,209515	173	2,238046	184	2,264818	195	2,290035
163	2,212188	174	2,240549	185	2,267172	196	2,292256
164	2,214844	175	2,243038	186	2,269513	197	2,294466
165	2,217484	176	2,243551	187	2,271842	198	2,296665
166	2,220108	177	2,247973	188	2,274158	199	2,298853
						200	2,301030

This Table has not been inserted so much with a view to practical use, as to illustrate the formation, and manner of using Logarithms.